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On the surface of superfluids

Jay Armas,^a Jyotirmoy Bhattacharya,^b Akash Jain^b and Nilay Kundu^c

^a*Physique Théorique et Mathématique Université Libre de Bruxelles and
International Solvay Institutes ULB,
Campus Plaine CP231, B-1050 Brussels, Belgium*

^b*Centre for Particle Theory & Department of Mathematical Sciences, Durham University,
South Road, Durham DH1 3LE, U.K.*

^c*Center for Gravitational Physics, Yukawa Institute for Theoretical Physics (YITP),
Kyoto University,
Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan*

E-mail: jarmas@ulb.ac.be, jyotirmoy.bhattacharya@durham.ac.uk,
akash.jain@durham.ac.uk, nilay.tifr@gmail.com

ABSTRACT: Developing on a recent work on localized bubbles of ordinary relativistic fluids, we study the comparatively richer leading order surface physics of relativistic superfluids, coupled to an arbitrary stationary background metric and gauge field in $3 + 1$ and $2 + 1$ dimensions. The analysis is performed with the help of a Euclidean effective action in one lower dimension, written in terms of the superfluid Goldstone mode, the shape-field (characterizing the surface of the superfluid bubble) and the background fields. We find new terms in the ideal order constitutive relations of the superfluid surface, in both the parity-even and parity-odd sectors, with the corresponding transport coefficients entirely fixed in terms of the first order bulk transport coefficients. Some bulk transport coefficients even enter and modify the surface thermodynamics. In the process, we also evaluate the stationary first order parity-odd bulk currents in $2 + 1$ dimensions, which follows from four independent terms in the superfluid effective action in that sector. In the second part of the paper, we extend our analysis to stationary surfaces in $3 + 1$ dimensional Galilean superfluids via the null reduction of null superfluids in $4 + 1$ dimensions. The ideal order constitutive relations in the Galilean case also exhibit some new terms similar to their relativistic counterparts. Finally, in the relativistic context, we turn on slow but arbitrary time dependence and answer some of the key questions regarding the time-dependent dynamics of the shape-field using the second law of thermodynamics. A linearized fluctuation analysis in $2 + 1$ dimensions about a toy equilibrium configuration reveals some new surface modes, including parity-odd ones. Our framework can be easily applied to model more general interfaces between distinct fluid-phases.

KEYWORDS: Effective Field Theories, Spontaneous Symmetry Breaking, Holography and condensed matter physics (AdS/CMT), Holography and quark-gluon plasmas

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1 Introduction and summary

Matter in the universe exists in diverse forms and very often its collective behaviour is so complex that its detailed microscopic description becomes intractable. Fortunately, in many situations of interest, the low-energy collective behaviour can be captured by an effective theory with a few degrees of freedom. A prominent example of such a finite temperature effective theory is hydrodynamics, where the description is provided in terms

of a few fluid variables in the long-wavelength approximation. In this effective description, the relevant microscopic information is conveniently packaged into the parameters of the theory, referred to as the transport coefficients.

The universal nature of this description has led to its applications in a diverse range of physical situations, ranging from neutron-stars, quark-gluon plasma to numerous condensed matter systems. Hence, this subject has had a long history and has been extremely well studied in the past. However, quite recently there has been a renewed interest in this area, particularly following the realization that there are some important lacunae in the structural aspects of the fluid equations that have been considered so far. It was understood that new transport coefficients must be incorporated in the effective theory in order to adequately describe certain physical situations.¹ In fact, one of the most interesting aspects of some of these newly discovered coefficients is their parity-odd nature — a possibility that has been largely ignored in the rich and classic literature on the subject.²

In regimes where the hydrodynamic approximation is applicable, it is often observed that the same underlying microscopic theory can exist in distinct macroscopic phases. In situations where two such phases coexist, they are separated by a dynamical interface (or surface). If we wish to provide an effective description for such scenarios, then the hydrodynamic description must be appropriately generalized in order to include the effects specific to such surfaces. Our main goal in this paper is to explore new surface properties, especially in the context of superfluids, focusing on the parity-odd effects.

For the case of ordinary relativistic space-filling fluids, the degrees of freedom include the fluid velocity u^μ , temperature T and chemical potential(s) μ corresponding to any global symmetries that the fluid may enjoy. In this paper, we will assume this global symmetry to be a U(1) symmetry. The equation of motion for these fluid fields are simply the conservation of the energy-momentum tensor and charge current, which in turn are expressed in terms of the fluid variables subjected to constitutive relations. The structure of the constitutive relations is determined based on symmetry principles and is severely constrained by the second law of thermodynamics [6].

In the case of superfluids, the U(1) symmetry is spontaneously broken and the phase of the order parameter ϕ serves as a massless Goldstone boson, which must be included in the low-energy effective description in addition to the ordinary fluid fields. In order to preserve gauge invariance, ϕ enters into the constitutive relations only through its gauge-covariant derivative, referred to as the superfluid velocity ξ_μ (see [6] for more details on the basics of superfluid dynamics). In the case of space-filling superfluids, the most general constitutive relations consistent with the second law of thermodynamics up to first order in the derivative expansion have been worked out more recently in [4].

If we wish to provide a unified description of two (super)fluid phases separated by a dynamical surface, we need to include a new field f in the hydrodynamic description,

¹For instance, in situations where a symmetry of the underlying theory suffers from an anomaly. This was first realized in the study of the map between long-wavelength fluctuation of black holes and relativistic hydrodynamics [1, 2]. Subsequently, the existence of such new coefficients has been inferred employing pure hydrodynamic techniques [3].

²See [4, 5] for recent discussions on such parity odd transport properties.

which keeps track of the shape of the surface. The surface is considered to be located at $f = 0$. This shape-field f is quite analogous to the Goldstone boson ϕ in the case of superfluids. In fact, f may be considered to be the Goldstone boson corresponding to the spontaneous breaking of translational invariance in the direction normal to the fluid surface.³ The guiding symmetry principle for incorporating this shape-field into the constitutive relations is the reparametrization invariance, i.e. the fluid must be invariant under arbitrary redefinitions of f as long as its zeroes are unchanged. This essentially implies that the dependence of the fluid currents on f happens primarily⁴ through n_μ , the normal vector to the surface, and its derivatives.

Now, for a superfluid bubble placed inside an ordinary fluid, there is a rich interplay between the Goldstone boson ϕ and the shape-field f on the surface of the superfluid bubble. In this paper, we study these surface effects and work out the ideal order surface currents for a superfluid.

This paper is organized as follows: in the remaining of this section we will give a detailed summary of the main points and techniques used in this paper. In section 2 we discuss stationary superfluid bubbles suspended in ordinary fluids in $3+1$ dimensions, and extend it to $2+1$ dimensions in section 3 (see the summary in section 1.1). Then in section 4, we discuss stationary Galilean superfluid bubbles using the technique of null superfluids [10], and use it to understand about the non-relativistic limit of surface phenomenon in superfluids (see the summary in section 1.2). Later in section 5, we turn on slow but arbitrary time dependence and study time-dependent dynamics of the shape-field f using the second law of thermodynamics as well as linearized fluctuations about an equilibrium configuration. We finish with some discussion in section 6. The paper has three appendices. In appendix A we discuss surface thermodynamics for $2+1$ dimensional superfluid bubbles. Then in appendix B, we give a generic derivation of the Young-Laplace equation for stationary superfluid bubbles, that determines the shape of the surface. Finally, in appendix C we collect some useful formulae and notations.

1.1 Stationary superfluid bubbles

To begin with, following [8], we shall mainly focus on stationary relativistic superfluid bubbles, which will enable us to employ the partition function techniques discussed in [11–13]. Our main objective here is to write down a Euclidean effective action for the Goldstone boson ϕ and the shape-field f in one lower dimension, from which the surface currents can be easily read off using a variational principle. One of our primary focuses in this analysis will be the parity violating terms. Therefore, we will separately discuss the cases of $3+1$ and $2+1$ dimensions,⁵ which have significantly different parity-odd structures.

³See [7, 8] for a relevant recent discussion in the stationary case and [9] for an application of similar ideas to the study of polarization effects on surface currents in the context of magnetohydrodynamics.

⁴As we shall explain in more detail below, another way in which f may enter the constitutive relations is via the distribution function $\theta(f)$ and its reparametrization invariant derivatives.

⁵Note that there is a subtlety in the discussion of finite temperature superfluidity in $2+1$ dimensions. At finite temperature, the low-energy physics is blind to the time-like direction and therefore the dynamics is effectively two dimensional. In our context, this is clearly reflected by the fact that in $2+1$ dimensions we write down a two dimensional Euclidean action for the massless Goldstone boson. This brings us within the purview of the Mermin-Wagner theorem implying that superfluidity in these dimensions may be destroyed

We will consider stationary bubbles of a superfluid in the most general background spacetime metric and gauge field which admits a time-like Killing vector ∂_t

$$\begin{aligned} ds^2 &= \mathcal{G}_{\mu\nu} dx^\mu dx^\nu = -e^{2\sigma(\vec{x})} (dt + a_i(\vec{x}) dx^i)^2 + g_{ij}(\vec{x}) dx^i dx^j, \\ \mathcal{A} &= \mathcal{A}_0(\vec{x}) dt + \mathcal{A}_i(\vec{x}) dx^i. \end{aligned} \quad (1.1)$$

Here, the i -index runs over the spatial coordinates. We will denote the covariant derivative associated with $\mathcal{G}_{\mu\nu}$ by ∇_μ , while the one associated with g_{ij} by D_i . For later use, we also define respective surface derivatives by $\tilde{\nabla}_\mu(\cdots) = 1/\sqrt{\nabla^\nu f \nabla_\nu f} \nabla_\mu(\sqrt{\nabla^\nu f \nabla_\nu f} \cdots)$ and the correspondent one associated to g_{ij} by $\tilde{D}_i(\cdots) = 1/\sqrt{D^j f D_j f} D_i(\sqrt{D^j f D_j f} \cdots)$. Now, since we wish to provide a finite temperature partition function⁶ description of our system, we will Wick-rotate to Euclidean time and compactify this direction, with an inverse radius T_0 . Thus, the set of all background data comprises of (see [8, 11] for more details)

$$\{\sigma(\vec{x}), a_i(\vec{x}), g_{ij}(\vec{x}), \mathcal{A}_0(\vec{x}), \mathcal{A}_i(\vec{x}), T_0\}. \quad (1.2)$$

Apart from T_0 , there is another length scale in the problem corresponding to the chemical potential μ_0 . However, it is always possible to absorb this into the time component of the arbitrary gauge field \mathcal{A}_0 . Therefore, we will not make μ_0 explicit in our discussions.

In addition to the background data (1.2), there are two fields which must be included in the partition function if we wish to describe superfluid bubbles. One of them is the phase of the scalar operator responsible for the spontaneous breaking of the U(1) symmetry, which we denote by $\phi(\vec{x})$ (see [12] for more details). The other is the shape-field $f(\vec{x})$, where $f(\vec{x}) = 0$ denotes the location of the interface between the superfluid and the ordinary charged fluid (see [8] for more details). In the superfluid description, the first derivative of the Goldstone boson $\phi(\vec{x})$ is treated as a quantity which is zeroth order in derivatives, and is referred to as the superfluid velocity $\xi_\mu = -\partial_\mu \phi + \mathcal{A}_\mu$.⁷ In the reduced language, that is, using the KK decomposition (1.1), since ϕ is time independent, we have that $\xi_\mu = \{\xi_0 = \mathcal{A}_0, \xi_i = -\partial_i \phi + \mathcal{A}_i\}$.

As has been explained in detail in [8, 11], the partition function must be constructed in terms of quantities that are invariant under spatial diffeomorphisms, Kaluza-Klein (KK) gauge transformations (redefinitions of time, $t \rightarrow t + \vartheta^t(\vec{x})$) and U(1) gauge transformations. Therefore, following [11], we first define a KK invariant gauge field

$$A = A_0 dt + A_i dx^i, \quad \text{where,} \quad A_0 \equiv \mathcal{A}_0, \quad \text{and,} \quad A_i \equiv \mathcal{A}_i - \mathcal{A}_0 a_i. \quad (1.3)$$

by strong quantum fluctuations. However, this conclusion is rendered invalid in the large-N limit. In fact, much of our discussions here might be relevant for 3 + 1 dimensional hairy black holes in AdS, via the AdS/CFT correspondence [14]. Also, our discussion in 2 + 1 dimensions may be relevant for other microscopic mechanisms for 2 + 1 dimensional superfluidity like the BKT transition. It would be definitely interesting to make this connection more precise.

⁶Here by partition function we refer to (the exponential of) the Euclidean effective action in the presence of arbitrary background sources.

⁷Here we follow the conventions of [12]. See also [4, 15] for an out of equilibrium discussion of relativistic superfluids.

In the context of superfluids, it is convenient to redefine the spatial components of the superfluid velocity so that they are invariant under both the U(1) and KK gauge transformations (the time component is automatically invariant) [12]

$$\zeta_i \equiv \xi_i - A_0 a_i = -\partial_i \phi + A_i. \quad (1.4)$$

The dependence of the partition function on the shape-field $f(\vec{x})$ follows exactly the same form as described in [8]. This dependence is primarily constrained by the reparametrization invariance of the surface $f \rightarrow g(f)$ with $g(0) = 0$. The elementary reparametrization invariant building block made out of f is the normal vector to the surface n_μ , which for stationary configurations takes the form

$$n_\mu = -\frac{\partial_\mu f}{\sqrt{\nabla_\nu f \nabla^\nu f}} = \{0, n_i\}, \quad \text{where,} \quad n_i = -\frac{\partial_i f}{\sqrt{D_j f D^j f}}. \quad (1.5)$$

We would again like to emphasize the remarkable similarity in the way the fields ϕ and f enter the partition function. In fact following the analogy, we will consider the normal vector n_μ , just like the superfluid velocity ξ_μ , as a zero derivative order quantity.

Now, we wish to describe a stationary bubble of a superfluid inside an ordinary charged fluid. The entire set of data which constitutes the building blocks of the partition function for ordinary charged fluids away from the interface are

$$\tilde{\mathbb{B}} = \{\sigma(\vec{x}), a_i(\vec{x}), g_{ij}(\vec{x}), A_0(\vec{x}), A_i(\vec{x}), T_0\}. \quad (1.6)$$

On the superfluid side, away from the interface, there is an additional ingredient

$$\mathbb{B} = \tilde{\mathbb{B}} \cup \{\zeta_i(\vec{x})\}. \quad (1.7)$$

On the surface, this set must also include the normal vector to the interface

$$\mathbb{B}_{(s)} = \mathbb{B} \cup \{n_i(\vec{x})\}. \quad (1.8)$$

The structure of the Euclidean effective action for a bubble of a superfluid inside a charged fluid will take the most general form

$$W = \int \{d\vec{x}\} \sqrt{g} \frac{e^\sigma}{T_0} \left(\theta(f) \mathcal{S}_{(b)}(\mathbb{B}, \partial\mathbb{B}, \dots) + \tilde{\delta}(f) \mathcal{S}_{(s)}(\mathbb{B}_{(s)}, \dots) + \theta(-f) \mathcal{S}_{(e)}(\tilde{\mathbb{B}}, \partial\tilde{\mathbb{B}}, \dots) \right), \quad (1.9)$$

where $\mathcal{S}_{(b)}$ and $\mathcal{S}_{(e)}$ are the partition functions of space-filling superfluids and ordinary charged fluids respectively, while $\mathcal{S}_{(s)}$ is the partition function of the interface. $\theta(f)$ is a distribution which captures the thickness of the wall. For an infinitely thin wall, $\theta(f)$ can be taken to be the Heaviside theta function. Furthermore, $\tilde{\delta}(f) = -n^\mu \partial_\mu \theta(f) = \sqrt{\nabla^\mu f \nabla_\mu f} \delta(f)$ (with $\delta(f) = d\theta(f)/df$) is the reparametrization invariant derivative of $\theta(f)$. Here, $\mathcal{S}_{(b)}, \mathcal{S}_{(e)}, \mathcal{S}_{(s)}$ are expanded in a derivative expansion as in ordinary fluid dynamics. In addition, one must also consider terms containing reparametrization invariant derivatives of $\tilde{\delta}(f)$ (i.e. terms with two or higher derivatives of $\theta(f)$). In this way, there are, in fact, two dimensionless small parameters in the effective theories studied in this paper.

One is the usual fluid expansion parameter $\omega/T \ll 1$ (ω being the typical frequency of fluctuations), which allows us to make the usual derivative expansion in fluid dynamics. The other small parameter is $\tau T \ll 1$ (τ being the length scale associated with the thickness of the surface (see [8] for more details)). The derivatives of $\theta(f)$ keep track of this second parameter. Thus, (1.9) should be thought of as a double expansion in both these parameters.

The energy-momentum tensor and charge current that follows from the partition function (2.1), have the structural form

$$\begin{aligned} T^{\mu\nu} &= \theta(f) T_{(b)}^{\mu\nu} + \tilde{\delta}(f) T_{(s)}^{\mu\nu} + \theta(-f) T_{(e)}^{\mu\nu} + \dots, \\ J^\mu &= \theta(f) J_{(b)}^\mu + \tilde{\delta}(f) J_{(s)}^\mu + \theta(-f) J_{(e)}^\mu + \dots, \end{aligned} \quad (1.10)$$

where the ellipsis denotes terms with higher derivatives of $\theta(f)$. We will refer to $\mathcal{S}_{(b)}$ as the bulk of the superfluid bubble, $\mathcal{S}_{(e)}$ as the exterior and $\mathcal{S}_{(s)}$ as the surface. Correspondingly, $T_{(b)}^{\mu\nu}$, $J_{(b)}^\mu$ are bulk superfluid currents, $T_{(e)}^{\mu\nu}$, $J_{(e)}^\mu$ are exterior fluid currents, and $T_{(s)}^{\mu\nu}$, $J_{(s)}^\mu$ are surface currents. The former two have been well explored in the literature (see e.g. [11, 12]), so our main focus here will be on the surface currents, and how the bulk/exterior of the bubble affects the surface. The conservation of energy-momentum tensor and charge current in (1.10), serve as the fluid equation of motion.

In this paper, we will obtain the surface currents in (1.10) in a special hydrodynamic frame,⁸ which is the frame that follows directly from equilibrium partition functions. In this frame, defining $\tilde{K} = \partial_t$ as the time-like Killing vector field of the background, the usual ordinary fluid variables are given by

$$T = e^{-\sigma} T_0, \quad u^\mu = e^{-\sigma} \tilde{K}^\mu, \quad \mu = e^{-\sigma} A_0, \quad (1.11)$$

to all orders in the derivative expansion. We will refer to this frame as the partition function frame. Furthermore, the surface equations of motion that follow from the conservation of the currents (1.10) can be thought of as equations which constrain the boundary conditions that should be imposed when solving the bulk equations. If we work in a regime where $\tilde{\delta}'(f)$ terms can be neglected, then the exercise of finding new configurations reduces to a boundary value problem from the bulk point of view. This problem should be solved with the boundary conditions themselves being determined by the surface conservation equations.⁹

Before summarizing our results regarding the detailed structure of the partition function, we would like to justify the distribution function $\theta(f)$ appearing in (1.9) in terms of the Landau-Ginzburg paradigm. Here, we are describing an interface between two phases, distinguished by the status of a U(1) symmetry, which is spontaneously broken in one phase, while is intact in the other. The two phases, therefore, are distinguished by an order parameter, with the help of which it is possible to write down a Landau-Ginzburg

⁸See [8] for a detailed and complete description of issues on the choice of frames.

⁹We would like to emphasize that using solutions of the surface equations as boundary conditions for the bulk equations is clearly consistent at least in equilibrium, where there exists a continuous solution (1.11) of the fluid variables for the combined set of equations, following from the conservation of currents in (1.10) (see [8] for more detailed discussion of this issue). In section 5, we also show that this method may be applied in time-dependent situations as well at leading order in derivatives.

theory

$$\mathcal{A}_{LG} = \int |\mathcal{D}_\mu \Phi|^2 + a|\Phi|^2 + b|\Phi|^4, \quad (1.12)$$

where $\Phi = \psi e^{i\phi}$ is a complex scalar field. Here ψ is the order parameter, which is 1 in the superfluid phase and is 0 outside, and smoothly interpolates between 1 and 0 on the interface separating the two phases. The hydrodynamic degrees of freedom can be seen as small fluctuations about the profile of the condensate. In such situations, the profile of ψ itself provides us with a smooth distribution function $\theta(f)$ required in the partition function (1.9). The terms that are proportional to the derivatives of ψ are localized on the interface and contribute to $\mathcal{S}_{(s)}$ in (1.9). Now, as we have discussed before, the derivative of the phase ϕ , referred to as the superfluid velocity, enters the superfluid dynamics. We would like to point out that ψ starts decreasing from 1, as we approach the interface from the superfluid side, and goes to zero with the onset of the ordinary charged fluid. This implies that it is possible to have a non-trivial profile of ϕ at the interface. This prompts us to include a dependence of the interface partition function $\mathcal{S}_{(s)}$ on the superfluid velocity. In this context, it is worthwhile pointing out that expanding around the background interpolating profile of ψ , and keeping terms up to the quadratic order in ϕ , we see that $\mathcal{S}_{(s)}$ can depend on the magnitude of the superfluid velocity, as well as on its component along the direction normal to the surface, both being Lorentz scalars from the interface point of view. Following the analogy with ordinary fluids,¹⁰ it is tempting to anticipate that the component of the superfluid velocity normal to the surface should vanish in the stationary case. However, we were unable to obtain any rigorous justification why this should be the case, and hence we will perform all our analyses keeping this component non-zero and arbitrary. In fact, an entropy current analysis at leading order, performed in section 5.1.2 for situations away from equilibrium, also allows for a non-zero component of the superfluid velocity normal to the surface.

It may also be noted that, while describing superfluids where there is a normal fluid component, the usual fluid fields (u^μ , T and μ) are also present along with Φ in the Landau-Ginzburg setting. All these fields, including Φ , may be composite or effective fields constituted out of the more fundamental degrees of freedom. In such situations, we may consider interaction terms between Φ and other fluid variables in the effective action (1.12). In 2+1 dimensions, in particular, it is possible to write down such an interesting parity-odd interaction term of the form

$$\mathcal{A}_{LG} = \int \cdots + i\epsilon^{\mu\nu\rho} u_\mu (\mathcal{D}_\nu \Phi) (\mathcal{D}_\rho \Phi)^*. \quad (1.13)$$

Again, considering fluctuations about the background interpolating ψ , it is evident that (1.13) generates a term of the form $\epsilon^{\mu\nu\rho} u_\mu n_\nu \xi_\rho$ localized on the interface. In a time-independent context, in the reduced language, this would imply that in general the surface partition function $\mathcal{S}_{(s)}$ can depend on $\bar{\lambda} = \epsilon^{ij} n_i \zeta_j$. As we will see later, this fact has an important and non-trivial consequence on the surface thermodynamics of 2+1 dimensional superfluid bubbles.

¹⁰For ordinary fluids, the normal component of the fluid velocity at the surface vanishes in equilibrium, i.e. $u^\mu n_\mu|_{f=0} = 0$. See [8] and section 5 for further details.

The construction of $S_{(b)}$, up to first order in derivative expansion in $3 + 1$ dimensions was presented in [12] and is given by¹¹

$$\begin{aligned}
 W &= W_{\text{even}} + W_{\text{odd}}, \quad \text{where} \\
 W_{\text{even}} &= \int d^3x \sqrt{g} \frac{e^\sigma}{T_0} \theta(f) \left(P_{(b)} - e^{-\sigma} \alpha_1 \zeta^i \partial_i \sigma + \frac{\alpha_2}{T_0} \zeta^i \partial_i A_0 - \alpha_3 \text{D}_i (e^\sigma \mathcal{F} \zeta^i) \right), \\
 W_{\text{odd}} &= \int d^3x \sqrt{g} \frac{e^\sigma}{T_0} \theta(f) \left(T_0 \alpha_4 \epsilon^{ijk} \zeta_i \partial_j a_k + \alpha_5 \epsilon^{ijk} \zeta_i \partial_j A_k \right).
 \end{aligned} \tag{1.15}$$

Note that, as demonstrated in [12], the term proportional to α_3 is the leading order equation of motion of ϕ in the bulk and its effects can be trivially removed by a shift of ϕ .¹² Therefore, for the sake of simplicity we set α_3 to zero in our analysis.

In this paper, we also construct $S_{(b)}$ in $2 + 1$ dimensions up to first order in derivatives in section 3. The parity-even sector is identical to that of (1.15), while the parity-odd sector is richer than its $3 + 1$ dimensional counterpart¹³

$$W_{\text{odd}} = \int d^2x \sqrt{g} \frac{e^\sigma}{T_0} \theta(f) \left(m_\omega \epsilon^{ij} \partial_i a_j + m_B \epsilon^{ij} \partial_i A_j + \beta_1 \epsilon^{ij} \zeta_i \partial_j \sigma + \beta_2 \epsilon^{ij} \zeta_i \partial_j A_0 \right). \tag{1.16}$$

The bulk currents that follow from (1.16) have not yet been analyzed in the literature, to the best of our knowledge. We perform this exercise in section 3.1. We find that there is a total of 35 relations among transport coefficients that are determined in terms of the four coefficients in (1.16) (in addition to the parity-even terms).

Since we are only considering terms up to the first order in derivatives on both the sides far away from the surface, it suffices to only consider a zeroth order term at the surface for $\mathcal{S}_{(s)}$. This is the surface tension term which was considered in [8]. Since we will be dealing with superfluids on one side of the interface, the surface tension can now also depend on the superfluid velocity.

In $3 + 1$ dimensions, we work out the ideal order surface currents in (2.17) by varying the partition function, with associated surface thermodynamics given in (2.20). Later in section 3, we work out the analogous surface currents in $2 + 1$ dimensions in (3.36), with respective thermodynamics given in (3.37), which also includes parity-odd effects. One of the most interesting features of our equilibrium analysis is the fact that the equation of motion for the shape-field f (the Young-Laplace equation) is identical to the normal component of the energy-momentum conservation equation at the surface. We rigorously argue in appendix B that this must continue to hold at all orders in the derivative expansion.

¹¹Our convention for the spatial Levi-Civita tensor in $2 + 1$ dimensions is $\epsilon^{12} = 1/\sqrt{g}$ while in $3 + 1$ dimensions is $\epsilon^{123} = 1/\sqrt{g}$. Thus, we have the following reductions on the time circle

$$u_\mu \epsilon^{\mu\nu\sigma} \rightarrow -\epsilon^{ij}, \quad u_\mu \epsilon^{\nu\sigma\rho\mu} \rightarrow \epsilon^{ijk}, \tag{1.14}$$

in $2 + 1$ and $3 + 1$ dimensions respectively.

¹²In the presence of surfaces, the effect of such ϕ shifts at the surface can be absorbed by a redefinition of the surface partition function.

¹³Note that these terms are also the parity-odd first order corrections on the surface of the $3 + 1$ dimensional superfluid bubble.

1.2 Non-relativistic stationary superfluid bubbles

In order to obtain an understanding of the non-relativistic limits of superfluid surface currents, in section 4 we study Galilean¹⁴ superfluids in 3+1 dimensions. For this analysis, we use the technique of null (super)fluids developed in [10, 16, 17], where it was realized that transport properties of a Galilean (super)fluid are in one-to-one correspondence with that of a relativistic system - null (super)fluid in one higher dimension. Here, the basic idea is that in order to obtain the most generic Galilean (super)fluid currents in 3+1 dimensions, we can start with a null (super)fluid on a null background¹⁵ in 4+1 dimensions, and then perform a null reduction on it [18, 19] (also see [20, 21] for some earlier application of null reductions in the context of fluid dynamics). The null reduction reduces the underlying Poincaré symmetry algebra of a null (super)fluid to the Bargmann symmetry algebra (Galilean algebra with a central extension with the mass operator) of a Galilean (super)fluid. Though we find the null-reduction prescription more useful for our purposes, it is worth mentioning that these Galilean results can also be obtained directly in a 3+1 dimensional Newton-Cartan setting following [22, 23] (see also [24]).

The equilibrium currents of a null (super)fluid can be obtained from a partition function written in terms of the background fields, Goldstone boson and the shape-field, in very much the same way as for the relativistic fluids discussed in section 1.1. There is however, one crucial new ingredient for null backgrounds: in addition to the time-like Killing vector \tilde{K} as in (1.1), null backgrounds also have a null Killing vector V . Choosing a set of coordinates $\{x^M\} = \{x^-, t, x^i\}$ such that $\tilde{K} = \partial_t$ and $V = \partial_-$, the most general metric and gauge field configurations respecting both Killing vectors are given as

$$\begin{aligned} ds^2 &= \mathcal{G}_{MN} dx^M dx^N = -2e^{\sigma(\vec{x})} (dt + a_i(\vec{x})dx^i) (dx^- - \mathcal{B}_0(\vec{x})dt - \mathcal{B}_i(\vec{x})dx^i) + g_{ij}(\vec{x})dx^i dx^j, \\ \mathcal{A} &= -dx^- + \mathcal{A}_0(\vec{x})dt + \mathcal{A}_i(\vec{x})dx^i, \end{aligned} \quad (1.17)$$

where all the introduced quantities are independent of the x^- and t coordinates. In torsionless Galilean/null spacetimes, in equilibrium, we must also have that $\partial_i \sigma = \partial_{[i} a_{j]} = 0$. However, while writing an equilibrium partition function, we will not require our background to be torsionless and will only impose it at the end of the computation (see [16] for details). As in the relativistic case, we would like to construct the partition function in terms of all the background data which are manifestly invariant under diffeomorphisms on the null background and gauge transformations. In order to do so, we need to consider the following invariant combinations (we refer the reader to [10] for more details regarding the transformation properties)

$$B_i = \mathcal{B}_i - a_i \mathcal{B}_0, \quad B_0 = \mathcal{B}_0, \quad A_i = \mathcal{A}_i - a_i \mathcal{A}_0 - B_i, \quad A_0 = \mathcal{A}_0. \quad (1.18)$$

¹⁴There is a subtle difference between Galilean and non-relativistic systems. As we will explain in more detail below, in the context of fluid dynamics, non-relativistic fluids are only a special class of the Galilean ones. Moreover, there can be other non-relativistic systems, such as Lifshitz systems with a dynamical exponent $z \neq 1, 2$, which are not Galilean.

¹⁵A null background is one which admits a null Killing vector V^M such that a component of the gauge field is fixed as $V^M \mathcal{A}_M = -1$. A null fluid is a fluid which couples to such a null background and the respective fluid velocity u^M is null instead of time-like. It is normalized so that $u^M V_M = -1$.

Since we are interested in superfluids, we also have the Goldstone boson ϕ , and as in section 1.1 its only gauge invariant combination is $\zeta_i = -\partial_i\phi + A_i$. The full superfluid velocity thus takes the form $\xi_M = \partial_M\phi + \mathcal{A}_M = \{-1, A_0, \zeta_i + a_i A_0 + B_i\}$.

Compared to [10], the additional ingredient in our discussion is the shape-field f , since eventually we are interested in the non-relativistic limit of the superfluid surface. The surface of the null superfluid needs to respect both the Killing vectors V and \tilde{K} , rendering it independent of x^- and t coordinates. Again, since f can only appear in the partition function in a reparametrization invariant fashion, the primary dependence on f comes through the normal vector $n_M = \{0, 0, n_i\}$ with n_i being again given by (1.5).

The background data invariant under all the required symmetries, in terms of which the partition function for bubbles of a null superfluid should be constructed, is given by ($1/T_0$ is the radius of the Euclidean time circle)

$$\mathbb{B} = \{\sigma(\vec{x}), a_i(\vec{x}), g_{ij}(\vec{x}), B_0(\vec{x}), B_i(\vec{x}), A_0(\vec{x}), A_i(\vec{x}), \zeta_i(\vec{x}), T_0\}, \quad \mathbb{B}_{(s)} = \mathbb{B} \cup \{n_i(\vec{x})\}. \quad (1.19)$$

Note that in this case the background data is clearly larger compared to the relativistic case, leading to more terms in the partition function at any given derivative order. This in turn implies that the Galilean fluid obtained after null reduction will in general have more transport coefficients than its relativistic counterpart. This is to be expected for a non-relativistic fluid as well, e.g. in the non-relativistic limit the energy of a relativistic fluid splits into a rest mass density part and the residual internal energy, hence increasing the count. Though this counting accounts for the extra coefficients at ideal order, there is no reason to believe that at higher orders as well such splitting will account for all the extra transport coefficients of a Galilean fluid.¹⁶ Therefore, the most generic non-relativistic fluid is, at best, a subset of the Galilean fluid discussed in this paper, exploration of which we leave for future work.

Finally, the equilibrium partition function for a 4+1 dimensional null superfluid bubble immersed in an ordinary fluid, up to first derivative order in the bulk and ideal order on the surface, can be written as

$$\begin{aligned} W = & \int d^3x \sqrt{g} \theta(f) \frac{e^\sigma}{T_0} \left(P_{(b)} - f_1 \zeta^i \partial_i \sigma + e^{-\sigma} f_2 \zeta^i \partial_i A_0 + e^{-\sigma} f_3 \zeta^i \partial_i B_0 \right. \\ & \left. + T_0 e^{-\sigma} (g_1 + g_2) \epsilon^{ijk} \zeta_i \partial_j B_k + T_0 e^{-\sigma} g_2 \epsilon^{ijk} \zeta_i \partial_j A_k + T_0 (g_1 e^{-\sigma} B_0 + g_2 e^{-\sigma} A_0 - g_3) \epsilon^{ijk} \zeta_i \partial_j a_k \right) \\ & + \int d^3x \sqrt{g} \tilde{\delta}(f) \frac{e^\sigma}{T_0} \mathcal{C} + \int d^3x \sqrt{g} \theta(-f) \frac{e^\sigma}{T_0} P_{(e)}. \end{aligned} \quad (1.20)$$

Note that there are no possible first order terms that we can write on the ordinary fluid side outside the bubble. All the transport coefficients are functions of the zeroth order background scalar data, while those on the surface have an additional dependence on $n_i \zeta^i$ as in the relativistic case. Note that in writing (1.20), we have ignored a total derivative term

¹⁶Furthermore, transport coefficients of a Galilean fluid have dependence on an extra scalar as opposed to a relativistic fluid, namely, the mass chemical potential. However, for any non-relativistic fluid obtained as a limit of a relativistic fluid, this dependence must be trivial.

in the bulk, which can be absorbed in the surface term and, similarly to the relativistic case, we have not considered a bulk term proportional to the zeroth order ϕ equation of motion.

Using the partition function (1.20) and the variational formulae (C.4), we can work out the currents for a $4 + 1$ dimensional null superfluid bubble, which we report in (4.8). Given this, it is straightforward to exploit the null isometry to perform a null reduction and get the surface currents for a Galilean superfluid (4.9). Even in this case, we find that the ideal order surface currents receive contributions from the bulk transport coefficients leading to different thermodynamics compared to the bulk.

1.3 Time dependent fluctuations of the surface

Having understood the nature of the surface currents in equilibrium, we proceeded and introduced a slow but arbitrary time dependence. Away from equilibrium, there is no variational principle that can help us in deducing the structure of surface currents.¹⁷ Therefore, we have to resort back to the second law of thermodynamics in order to constrain the transport coefficients.

The surface of the fluid interacts freely with the bulk. In order to account for this exchange of degrees of freedom between the bulk and the surface, the local form of the second law at the surface needs to be suitably modified. This modification takes the following form

$$\tilde{\nabla}_\mu J_{(s)\text{ent}}^\mu - n_\mu J_{(b)\text{ent}}^\mu \geq 0, \quad (1.21)$$

where $J_{(s)\text{ent}}^\mu$ and $J_{(b)\text{ent}}^\mu$ represent the local surface and bulk entropy currents respectively. Eq. (1.21) corresponds to the $\tilde{\delta}(f)$ equation obtained from the divergence of the total entropy current which is of the form¹⁸

$$J_{\text{ent}}^\mu = J_{(b)\text{ent}}^\mu \theta(f) + J_{(s)\text{ent}}^\mu \tilde{\delta}(f) + \dots \quad (1.22)$$

There are a few important aspects of out of equilibrium dynamics that are a priori unclear, even in the context of ordinary fluid surfaces. One of the key aspects that needs to be understood is the nature of the normal component of the fluid velocity $u_\mu n^\mu$ at the surface. In equilibrium, $u_\mu n^\mu$ vanishes by construction but once the location of the surface becomes time dependent, this component may become non-trivial. Drawing from the analogy between f and ϕ , this problem is analogous to the problem of understanding how the Josephson equation $u^\mu \xi_\mu = \mu + \mu_{\text{diss}}$ is determined. In a recent paper [27], it was observed that the Josephson equation, even at ideal order, followed from the second law of thermodynamics, when the ϕ field is considered off-shell. While working in an appropriate generalization of the partition function frame, μ_{diss} reduces to the equation of motion of ϕ

¹⁷Given some of the latest developments in writing down actions in terms of fluid variables in non-equilibrium situations [25, 26], it would be interesting to understand if this setup can be suitably generalized to describe out of equilibrium fluid surfaces as well.

¹⁸The reader may wonder, since the second law is expressed as an inequality for the divergence of the total entropy current, whether it is legitimate to implement the inequality separately for terms proportional to $\theta(f)$ and $\delta(f)$. This is, however justified, since there can be fluid configurations where a non-trivial bulk entropy current is divergence free and the second law inequality must be valid for all fluid configurations.

in equilibrium. This leads to the interpretation of the Josephson condition as the equation of motion of ϕ away from equilibrium.

Following this analogy, we consider $u^\mu n_\mu = \gamma + \gamma_{\text{diss}}$. In section 5.1.1, we demonstrate that the local form of the second law of thermodynamics on the surface sets γ to zero. The form of γ_{diss} is frame dependent like the Josephson condition. We derive γ_{diss} in a frame which is the appropriate generalization of the partition function frame in (1.11). In equilibrium, it reduces to the equation of motion of f (or equivalently to the Young-Laplace equation, which is the component of the energy-momentum conservation equation normal to the surface). Also, it is noteworthy that in out of equilibrium situations, the equation $u^\mu n_\mu = \gamma_{\text{diss}}$ is distinct from the corresponding Young-Laplace equation. They together determine two scalar degrees of freedom at the boundary: $u^\mu n_\mu$ and f , the former of which turns out to be trivial in equilibrium.

Proceeding to the superfluid case in out of equilibrium scenarios, we tackle the corresponding problem for the normal component of the superfluid velocity at the surface $n^\mu \xi_\mu = \lambda + \lambda_{\text{diss}}$. In equilibrium, the ϕ equation of motion at $\tilde{\delta}'(f)$ order imposes the condition that $\partial\mathcal{C}/\partial\lambda = 0$, where \mathcal{C} is (minus) the surface tension. Given a particular dependence of \mathcal{C} on λ , implied by some microscopic description, the condition $\partial\mathcal{C}/\partial\lambda = 0$ should be seen as the equation determining the value of λ at the surface. However, in the special case for which the surface tension does not depend on λ , the effective action does not impose any restriction on λ . The equation of motion for ϕ may be obtained by an off-shell implementation of the second law, as in [27]. However, in the case of λ , we have shown in section 5.1.2 that an entropy current analysis does not impose any constraints on λ , once the leading order entropy density is modified by terms involving λ . This modification to the entropy density is identical to what is obtained from the equilibrium partition function in section 2.2. Since none of the physical constraints is able to set λ to zero, we report all our results keeping λ arbitrary. Note that there can of course be many configurations with $\lambda = 0$ but our analysis suggests that these will only be a subset of all possible configurations.

Also, as explained previously, the surface equations may also be interpreted as determining the possible set of boundary conditions that are allowed for the bulk fluid equations. Clearly, in the equilibrium case, there are consistent solutions to the full set of bulk and surface equations. In the partition function frame, such a solution corresponded to the one where the fluid velocity is aligned with a Killing vector field of the background. However, away from equilibrium, even with a judicious choice of frame, such a solution may be considerably complicated. In order to obtain some idea of the nature of such solutions in time-dependent cases, we study the linearized fluctuations around a toy equilibrium configuration, only considering the perfect fluid equations of motion.

In section 5.2, we work with 2+1 dimensional ordinary fluids in flat space and consider the background equilibrium configuration to be one in which a static fluid fills half space. At first, we set the surface entropy to zero, recovering the standard dispersion relation of surface capillary waves $\omega \sim \pm k^{3/2}$. If the amplitude of the surface ripples is much larger than the surface thickness, then ignoring the surface degrees of freedom is a perfectly legitimate approximation. However, as soon as we allow the surface tension to be a function of T , thus introducing some non-trivial surface entropy, our surface equations predict a dispersion relation of the form $\omega \sim \pm k$. We are then able to solve the bulk equations with

such sound-like boundary conditions. This new kind of surface sound wave for ordinary fluids is expected to be visible if the amplitude of the waves is comparable or less than the surface thickness. These waves are very reminiscent of the third sound mode for superfluids. We perform a similar analysis for 2+1 dimensional superfluids, for which the leading order surface equations contain parity-odd terms. We find that parity violation leaves its imprint on the spectrum of linearized fluctuations, which contains a sound mode with $\omega \sim k$ while its partner under a parity transformation $k \rightarrow -k$ is absent.

2 Stationary superfluid bubbles in 3+1 dimensions

In this section we study stationary bubbles of a 3 + 1 dimensional relativistic superfluid immersed in an ordinary fluid. We work out the respective constitutive relations up to first derivative order in the bulk and ideal order at the interface using equilibrium partition functions.

2.1 Perfect superfluid bubbles (d+1 dimensions, d ≥ 3)

A discussion of the surface properties in perfect superfluids was initiated in [8]. Here we will elaborate and extend upon that discussion. As explained in section 1.1, the equilibrium partition function for superfluids takes the form given in (1.9). If the partition function does not contain any derivatives, the respective superfluid is called a perfect superfluid. It is of course a fictitious simplified system just like a perfect fluid, nevertheless it is an instructive toy system to study before moving to more complicated generalizations.

For a perfect superfluid bubble with an ordinary charged fluid outside, the most generic partition function takes the form

$$W = \int d^3x \sqrt{g} \frac{e^{-\sigma}}{T_0} \left(\theta(f) P_{(b)}(T, \mu, \chi) + \tilde{\delta}(f) \mathcal{C}(T, \mu, \tilde{\chi}, \lambda) + \theta(-f) P_{(e)}(T, \mu) \right), \quad (2.1)$$

where we have defined $T = T_0 e^{-\sigma}$ and $\mu = A_0 e^{-\sigma}$ suggestively for later identification with the temperature and chemical potential respectively, while $\lambda = n_\mu \xi^\mu = n_i \zeta^i$, $\chi = -\xi^\mu \xi_\mu = \mu^2 - \zeta^i \zeta_i$, $\tilde{\xi}^\mu = -(\mathcal{G}^{\mu\nu} - n^\mu n^\nu) \xi_\nu$ and $\tilde{\chi} = -\tilde{\xi}^\mu \tilde{\xi}_\mu = \mu^2 - \zeta^i \zeta_i + \lambda^2$. As we will see later, $P_{(b)}$ and $P_{(e)}$ are the bulk and external pressures while \mathcal{C} will be identified as the negative of surface tension. The discussion in this subsection is immediately applicable to perfect superfluid bubbles in all dimensions, except in 2+1 dimensions¹⁹ where there can be parity-odd effects at ideal order, and will be treated separately in section 3.

We start by varying the partition function (2.1) with respect to the Goldstone boson ϕ , and work out the respective equations of motion

$$\begin{aligned} \theta(f) \left[D_i \left(\frac{2}{T} \frac{\partial P_{(b)}}{\partial \chi} \zeta^i \right) \right] &= 0, \\ \tilde{\delta}(f) \left[T \tilde{D}_i \left(\frac{1}{T} \frac{\partial \mathcal{C}}{\partial \lambda} n^i - \frac{2}{T} \frac{\partial \mathcal{C}}{\partial \tilde{\chi}} \tilde{\zeta}^i \right) + 2\lambda \frac{\partial P_{(b)}}{\partial \chi} \right] &= 0, \\ \tilde{\delta}'(f) \left[\frac{\partial \mathcal{C}}{\partial \lambda} \right] &= 0, \end{aligned} \quad (2.2)$$

¹⁹Even in 2+1 dimensions, if we restrict only to the parity even sector, then the discussion of this section is applicable.

where D_i denotes the spatial covariant derivative associated with g_{ij} , while \tilde{D}_i denotes the spatial covariant derivative on the surface defined in section 1.1. The last line of this equation is particularly interesting, as it tells us that on-shell, the boundary function \mathcal{C} is independent of the component of the superfluid velocity along n_μ , i.e. $\lambda = n^\mu \xi_\mu$,²⁰ and is only dependent on the projected components $\tilde{\xi}_\mu$ through $\tilde{\chi}$.

The first line in (2.2) is a non-linear second order differential equation, which yields the profile of the Goldstone mode ϕ in the bulk of the superfluid bubble. For cases where the superfluid velocity can be taken to be small, this equation may be linearized and converted into a second order linear partial differential equation. This equation must be solved with suitable boundary conditions at the interface, which are provided by the solutions to the second and third lines in (2.2). The third equation provides the derivative of ϕ normal to the interface, while the second equation provides the initial condition necessary to evolve the first equation away from the interface. Note that as we move to higher orders, we will have an additional condition at the surface, and correspondingly, the order of the first differential equation will increase by one.

Varying the partition function (2.1) and using the variational formulae (C.3), we can read out the bulk and boundary currents. The form of the energy-momentum tensor and charge current inside the bubble takes the usual perfect superfluid form and has been thoroughly discussed in [12], while outside the bubble it is just an ordinary perfect charged fluid. The new ingredients in our discussion however are the currents at the interface, found via variation as (upon using the $\tilde{\delta}'(f)$ order ϕ equation of motion)

$$\begin{aligned} T_{(s)00} &= e^{2\sigma} \left(-\mathcal{C} + T \frac{\partial \mathcal{C}}{\partial T} + \mu \frac{\partial \mathcal{C}}{\partial \mu} \right) + 2\xi_0^2 \frac{\partial \mathcal{C}}{\partial \tilde{\chi}}, & T_{(s)}^i{}_0 &= \xi_0 2 \frac{\partial \mathcal{C}}{\partial \tilde{\chi}} \tilde{\zeta}^i, & T_{(s)}^{ij} &= \mathcal{C} h^{ij} + 2 \frac{\partial \mathcal{C}}{\partial \tilde{\chi}} \tilde{\zeta}^i \tilde{\zeta}^j, \\ J_{(s)0} &= -e^\sigma \frac{\partial \mathcal{C}}{\partial \mu} - 2\xi_0 \frac{\partial \mathcal{C}}{\partial \tilde{\chi}}, & J_{(s)}^i &= -2 \frac{\partial \mathcal{C}}{\partial \tilde{\chi}} \tilde{\zeta}^i, \end{aligned} \quad (2.3)$$

where $h^{ij} = g^{ij} - n^i n^j$ and $\tilde{\zeta}^i = h^{ij} \zeta_j$. Thus we see that, just as in the bulk of the superfluid, there is energy and charge transport along the superfluid velocity, also on the surface.

It is further instructive to write down the equation for the shape-field f that follows from the partition function (2.1) (upon using the $\tilde{\delta}'(f)$ order ϕ equation of motion)

$$P_{(b)} - P_{(e)} + T D_i \left(\frac{1}{T} \mathcal{C} n^i + \frac{2\lambda}{T} \frac{\partial \mathcal{C}}{\partial \tilde{\chi}} \tilde{\zeta}^i \right) = 0. \quad (2.4)$$

This is the modified Young-Laplace equation in the present case. As argued in appendix B, this equation is simply the normal component of the energy-momentum conservation equation on the surface.

Let us now study the implications of this analysis on the covariant form of the charge current and energy-momentum tensor. We would like to work in a hydrodynamic frame most suitable for the analysis using the partition function. It is a frame where we have

$$u^\mu = e^{-\sigma} (1, 0, 0, 0), \quad T = e^{-\sigma} T_0, \quad \mu = e^{-\sigma} A_0, \quad (2.5)$$

²⁰Since the surface tension function \mathcal{C} is given a priori, derived from the microscopics, the condition $\partial \mathcal{C} / \partial \lambda = 0$ should be thought of as a condition determining λ and not as a consistency condition on \mathcal{C} .

everywhere to all derivative orders, including at the interface. Such frame choice should be always possible to make as long as we are in equilibrium. The most general ideal order surface currents with the conditions that $T_{(s)}^{\mu\nu}n_\nu = J_{(s)}^\mu n_\mu = 0$,²¹ can be written as

$$\begin{aligned} T_{(s)}^{\mu\nu} &= (\mathcal{E} - \mathcal{Y})u^\mu u^\nu - \mathcal{Y}(\mathcal{G}^{\mu\nu} - n^\mu n^\nu) + \mathcal{F}\tilde{\xi}^\mu \tilde{\xi}^\nu + 2\lambda_1 u^{(\mu} \tilde{\xi}^{\nu)} + 2\lambda_2 \xi^{(\mu} \bar{n}^{\nu)} + 2\mathcal{U}u^{(\mu} \bar{n}^{\nu)}, \\ J_{(s)}^\mu &= \mathcal{Q}u^\mu + \mathcal{F}'\tilde{\xi}^\mu + \mathcal{V}\bar{n}^\mu, \end{aligned} \quad (2.6)$$

where we have defined $\bar{n}^\mu = \epsilon^{\mu\nu\rho\sigma}u_\nu \xi_\rho n_\sigma$ as the only parity-odd ideal order data. Now reducing (2.6) on the time circle and comparing it with (2.3), we obtain²²

$$\mathcal{E} = -\mathcal{C} + T\frac{\partial\mathcal{C}}{\partial T} + \mu\frac{\partial\mathcal{C}}{\partial\mu}, \quad \mathcal{Y} = -\mathcal{C}, \quad \mathcal{Q} = \frac{\partial\mathcal{C}}{\partial\mu}, \quad \mathcal{F} = -\mathcal{F}' = 2\frac{\partial\mathcal{C}}{\partial\tilde{\chi}}, \quad \mathcal{S} = \frac{\partial\mathcal{C}}{\partial T}, \quad (2.7)$$

and $\lambda_1 = \lambda_2 = \mathcal{U} = \mathcal{V} = 0$. Here, \mathcal{Y} is the surface tension and \mathcal{E} , \mathcal{Q} , \mathcal{S} are respectively the surface energy, charge and entropy densities, while \mathcal{F} is the surface superfluid density. We will see, however, that the coefficients \mathcal{U} , \mathcal{V} get non-zero values when we introduce first order terms in the bulk. The coefficient \mathcal{S} introduced in (2.7) is the surface entropy density and enters in the respective entropy current as $J_{(s)\text{ent}}^\mu = \mathcal{S}u^\mu$. From (2.7), we can now recover the Euler relation and the Gibbs-Duhem relation of thermodynamics respectively on the surface (upon using the $\tilde{\delta}'(f)$ order ϕ equation of motion)

$$\mathcal{E} - \mathcal{Y} = T\mathcal{S} + \mu\mathcal{Q}, \quad d\mathcal{Y} = -\mathcal{S}dT - \mathcal{Q}d\mu - \frac{1}{2}\mathcal{F}d\tilde{\chi}. \quad (2.8)$$

The first law of thermodynamics trivially follows from here as

$$d\mathcal{E} = Td\mathcal{S} + \mu d\mathcal{Q} - \frac{1}{2}\mathcal{F}d\tilde{\chi}. \quad (2.9)$$

These thermodynamic relations are exactly the same as their bulk counterparts. However, as we will show in the next subsection, the surface thermodynamics will modify upon including first order corrections in the bulk.

2.2 First order corrections away from the interface

Since the surface currents sit on a boundary separating two phases of a fluid, transport coefficients at a particular derivative order in the bulk can affect the surface currents at lower orders via an “inflow” (via a differentiation by parts in the partition function language). Therefore, we expect the ideal order surface currents to get contributions from first order terms in the bulk. In order to do so, we consider first order corrections to the bulk superfluid partition function (discussed in [12])

$$W^{(1)} = W_{\text{even}}^{(1)} + W_{\text{odd}}^{(1)}, \quad (2.10)$$

²¹The tangentiality conditions on the surface energy-momentum tensor and currents are a direct consequence of their conservation equations to leading order in the surface thickness. If thickness corrections are taken into account (by including $\tilde{\delta}'(f)$ terms or of higher derivative order in the energy-momentum tensor or currents) then these tangentiality conditions will be modified by extra terms on the right hand side [28–30].

²²Note that we are defining $\mathcal{Y} = -\mathcal{C}$ in order to have the same sign convention for the surface tension \mathcal{Y} as in classical literature.

where

$$\begin{aligned}
 W_{\text{even}}^{(1)} &= \int d^3x \sqrt{g} \frac{e^\sigma}{T_0} \theta(f) \left(\alpha_1 \zeta^i \partial_i T + \alpha_2 \zeta^i \partial_i \nu - \alpha_3 D_i \left(\frac{1}{T} \frac{\partial \mathcal{P}_{(b)}}{\partial \chi} \zeta^i \right) \right), \\
 W_{\text{odd}}^{(1)} &= \int d^3x \sqrt{g} \frac{e^\sigma}{T_0} \theta(f) \left(T_0 \alpha_4 \epsilon^{ijk} \zeta_i \partial_j a_k + \alpha_5 \epsilon^{ijk} \zeta_i \partial_j A_k \right).
 \end{aligned} \tag{2.11}$$

As discussed in section 1.1, while working up to first order in derivatives in the bulk of the superfluid, it is consistent to consider only the ideal order surface tension term at the surface, which was considered in (2.1). Also, far outside the superfluid bubble, the ordinary charged fluid does not receive any first order corrections, as there are no possible terms that can be written in the partition function. Consequently, $W^{(1)}$ in (2.11) constitutes the entire first order corrections to the perfect fluid partition function in (2.1).

The bulk energy-momentum tensor and charge current that follow form (2.11) have been thoroughly examined in [12]. In particular, it was pointed out in [12] that the term proportional to α_3 enters the constitutive relations in a trivial fashion. The reason is that, since α_3 multiplies the lower order equation of motion of ϕ , it can be shifted to zero by a suitable field redefinition of ϕ .²³ In the presence of a surface, such a shift would also involve surface quantities. However, at the level of the partition function for instance, we can always redefine the surface tension to absorb these terms and ignore any higher order terms.

The surface energy-momentum tensor and charge current, in addition to (2.3), will now also have the following contributions from (2.11) (after setting $\alpha_3 = 0$)

$$\begin{aligned}
 T_{(s)00} &= e^{2\sigma} \lambda T \alpha_1, \quad T_{(s)}^i{}_0 = e^\sigma (T \alpha_4 - \mu \alpha_5) \bar{n}^i, \quad T_{(s)}^{ij} = 0, \\
 J_{(s)0} &= -e^\sigma \lambda \frac{\alpha_2}{T}, \quad J_{(s)}^i = \alpha_5 \bar{n}^i,
 \end{aligned} \tag{2.12}$$

where $\bar{n}^i = \epsilon^{ijk} \zeta_j n_k$. The equation of motion of ϕ is modified to

$$\begin{aligned}
 \theta(f) \left[D_i \left(\frac{2}{T} \frac{\partial \mathcal{P}_{(b)}}{\partial \chi} \zeta^i + \frac{2}{T} \frac{\partial \alpha_1}{\partial \chi} \zeta^i \zeta^j \partial_j T - \frac{\alpha_1}{T} g^{ij} \partial_j T + \frac{2}{T} \frac{\partial \alpha_2}{\partial \chi} \zeta^i \zeta^j \partial_j \nu - \frac{\alpha_2}{T} g^{ij} \partial_j \nu \right. \right. \\
 \left. \left. + 2e^\sigma \frac{\partial \alpha_4}{\partial \chi} \zeta^i \epsilon^{ajk} \zeta_a \partial_j a_k - e^\sigma \alpha_4 \epsilon^{ijk} \partial_j a_k + \frac{2}{T} \frac{\partial \alpha_5}{\partial \chi} \zeta^i \epsilon^{ajk} \zeta_a \partial_j A_k - \frac{\alpha_5}{T} \epsilon^{ijk} \partial_j A_k \right) \right] = 0, \\
 \tilde{\delta}(f) \left[T \tilde{D}_i \left(\frac{1}{T} \frac{\partial \mathcal{C}}{\partial \lambda} n^i - \frac{2}{T} \frac{\partial \mathcal{C}}{\partial \chi} \tilde{\zeta}^i \right) + 2\lambda \frac{\partial \mathcal{P}_{(b)}}{\partial \chi} + 2\lambda \frac{\partial \alpha_1}{\partial \chi} \zeta^j \partial_j T - \alpha_1 n^j \partial_j T \right. \\
 \left. + 2\lambda \frac{\partial \alpha_2}{\partial \chi} \zeta^j \partial_j \nu - \alpha_2 n^j \partial_j \nu + 2\lambda T_0 \frac{\partial \alpha_4}{\partial \chi} \epsilon^{ajk} \zeta_a \partial_j a_k \right. \\
 \left. - T_0 \alpha_4 \epsilon^{ijk} n_i \partial_j a_k + 2\lambda \frac{\partial \alpha_5}{\partial \chi} \epsilon^{ajk} \zeta_a \partial_j A_k - \alpha_5 \epsilon^{ijk} n_i \partial_j A_k \right] = 0, \\
 \tilde{\delta}'(f) \left[\frac{\partial \mathcal{C}}{\partial \lambda} \right] = 0,
 \end{aligned} \tag{2.13}$$

²³We need to shift ϕ by a term proportional to α_3 . Note that in the natural way of counting derivatives for superfluids, ϕ is -1 order, while $\alpha_3(T, \nu, \chi)$ is zeroth order. Therefore, this entails shifting ϕ (and hence the superfluid velocity ξ_μ) by a higher order term.

while the modified f equation of motion (Young-Laplace equation) is given as (see appendix B for a detailed discussion on Young-Laplace equations in the generic case)

$$\mathcal{P}_{(b)} - \mathcal{P}_{(e)} + \alpha_1 \zeta^i \partial_i T + \alpha_2 \zeta^i \partial_i \nu + T_0 \alpha_4 \epsilon^{ijk} \zeta_i \partial_j a_k + \alpha_5 \epsilon^{ijk} \zeta_i \partial_j A_k + T D_i \left(\frac{1}{T} \mathcal{C} n^i + \frac{2\lambda}{T} \frac{\partial \mathcal{C}}{\partial \tilde{\chi}} \tilde{\xi}^i \right) = 0. \quad (2.14)$$

It is worth pointing out that, instead of the partition function W in (2.11), we could have started with a covariant version (ignoring the α_3 term), i.e.,

$$\begin{aligned} W_{\text{even}}^{(1)} &= \int d^4 x \sqrt{-\mathcal{G}} \theta(f) \left(\frac{f_1}{T} \zeta^\mu \partial_\mu T + T f_2 \zeta^\mu \partial_\mu \nu \right), \\ W_{\text{odd}}^{(1)} &= \int d^4 x \sqrt{-\mathcal{G}} \theta(f) \left(g_1 \epsilon^{\mu\nu\rho\sigma} \zeta_\mu u_\nu \omega_{\rho\sigma} + \frac{1}{2} g_2 \epsilon^{\mu\nu\rho\sigma} \zeta_\mu u_\nu F_{\rho\sigma} \right). \end{aligned} \quad (2.15)$$

Comparing it to (2.11), we can simply read out the respective coefficients

$$\alpha_1 = \frac{f_1}{T}, \quad \alpha_2 = f_2 T, \quad -T(\alpha_4 - \nu \alpha_5) = g_1, \quad \alpha_5 = g_2. \quad (2.16)$$

Now, the covariant form of the energy-momentum tensor and charge current, after imposing the equation of motion for ϕ , are modified from that in (2.6) to

$$\begin{aligned} T_{(s)}^{\mu\nu} &= \mathcal{E} u^\mu u^\nu + \mathcal{Y} \mathcal{P}^{\mu\nu} + \mathcal{F} \tilde{\xi}^\mu \tilde{\xi}^\nu + 2\mathcal{U} u^{(\mu} \bar{n}^{\nu)}, \\ J_{(s)}^\mu &= \mathcal{Q} u^\mu - \mathcal{F} \tilde{\xi}^\mu + \mathcal{V} \bar{n}^\mu, \end{aligned} \quad (2.17)$$

where $\bar{n}^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \xi_\rho n_\sigma$. Most notably, the coefficients of the surface currents now receive contributions from the first order transport coefficients and we have

$$\begin{aligned} \mathcal{E} &= -\mathcal{C} + T \frac{\partial \mathcal{C}}{\partial T} + \mu \frac{\partial \mathcal{C}}{\partial \mu} + \lambda f_1, & \mathcal{Y} &= -\mathcal{C}, & \mathcal{S} &= \frac{\partial \mathcal{C}}{\partial T} + \frac{\lambda}{T} (f_1 - \mu f_2), \\ \mathcal{Q} &= \frac{\partial \mathcal{C}}{\partial \mu} + \lambda f_2, & \mathcal{F} &= 2 \frac{\partial \mathcal{C}}{\partial \tilde{\chi}}, & \mathcal{U} &= g_1, & \mathcal{V} &= g_2. \end{aligned} \quad (2.18)$$

Here we have defined \mathcal{S} as the surface entropy density with the respective entropy current given by

$$J_{(s)\text{ent}}^\mu = \mathcal{S} u^\mu + \frac{1}{T} (\mathcal{U} - \mu \mathcal{V}) \bar{n}^\mu. \quad (2.19)$$

The identification (2.18) leads to the Euler relation and a modified Gibbs-Duhem relation of thermodynamics at the surface²⁴

$$\mathcal{E} - \mathcal{Y} = T\mathcal{S} + \mu\mathcal{Q}, \quad d\mathcal{Y} = -\left(\mathcal{S} - \frac{\lambda}{T} (f_1 - \mu f_2) \right) dT - \left(\mathcal{Q} - \lambda f_2 \right) d\mu - \frac{1}{2} \mathcal{F} d\tilde{\chi}. \quad (2.20)$$

²⁴In (2.20), f_1 and f_2 should be thought of as first order bulk transport coefficients (see [12]) rather than parameters of the partition function.

We clearly see that the thermodynamics is different from usual.²⁵ The respective modified first law of thermodynamics now takes the form

$$d(\mathcal{E} - \lambda f_1) = T d\left(\mathcal{S} - \frac{\lambda}{T}(f_1 - \mu f_2)\right) + \mu d(\mathcal{Q} - \lambda f_2) - \frac{1}{2}\mathcal{F}d\tilde{\chi}. \quad (2.21)$$

This modification can be interpreted as follows. The surface densities \mathcal{E} , \mathcal{Q} , \mathcal{S} and \mathcal{F} have, in general, two contributions: from the thermodynamics on the surface and from the inflow from the bulk. If we identify the inflow contributions to \mathcal{E} , \mathcal{Q} , \mathcal{S} and \mathcal{F} as λf_1 , λf_2 , $\lambda(f_1 - \mu f_2)/T$ and 0 respectively, the remaining thermodynamic contributions satisfy the thermodynamics (2.20)–(2.21).

Note that the parity-odd ideal order surface transport coefficients \mathcal{U} and \mathcal{V} (or correspondingly g_1 and g_2) do not enter the thermodynamics (2.20)–(2.21). However, since all the first order bulk transport coefficients f_1 , f_2 , g_1 , g_2 do modify the ideal order surface transport, they can be measured by carefully designing experiments which probe the ideal order surface properties of superfluids.

3 Stationary superfluid bubbles in 2+1 dimensions

In this section, we study stationary superfluid bubbles in 2+1 dimensions and particularly focus on the parity-odd sector, where there is a significant difference compared to the 3+1 dimensional case. In fact, an exhaustive analysis of the first order parity-odd terms in the bulk of 2+1 dimensional superfluids has not been executed so far, to the best of our knowledge. Therefore, we also evaluate the stationary bulk currents following from the parity-odd first order bulk partition function in section 3.1 before analyzing their surface effects.

3.1 Parity-odd effects for perfect superfluid bubbles

We have discussed perfect superfluids in general dimensions in section 2.1. However, as explained in section 1.1, in 2+1 dimensions there can be parity-odd terms which may have a non-trivial effect on the surface tension. Hence, before going into the details of the surface effects of first order corrections in the bulk of 2+1 dimensional superfluids, we will revisit the zeroth order case once more.

In 2+1 dimensions, apart from λ , it is also possible to define a parity-odd zeroth order scalar on the surface $\bar{\lambda} = \epsilon^{\mu\nu\sigma} n_\mu u_\nu \xi_\sigma = \epsilon^{ij} n_i \zeta_j$. As explained in section 1.1, due to the possible presence of a term like (1.13) in a Landau-Ginzburg effective theory, the surface

²⁵Note that the difference here is not just a mere matter of definition of \mathcal{S} and \mathcal{Q} . We could have simply defined $\mathcal{S} = \frac{\partial \mathcal{Y}}{\partial T}$ and $\mathcal{Q} = \frac{\partial \mathcal{Y}}{\partial \mu}$, in the usual thermodynamic fashion, but that would result in new terms in the Gibbs-Duhem relation, which would then become different from usual. Note that our definition of the charge \mathcal{Q} , for instance, corresponds to the quantity which is the zeroth order value of the surface charge current, projected along the direction of the surface fluid velocity. In the usual case of bulk thermodynamics, these two definitions of charge density would coincide, but not for the surface thermodynamics. This is because, the surface current contains additional zeroth order terms proportional to bulk first order transport coefficients.

tension will, in general, depend on $\bar{\lambda}$ and we can write²⁶

$$W = \int d^2x \sqrt{g} \frac{e^\sigma}{T_0} \left(\theta(f) P_{(b)}(T, \mu, \chi) + \tilde{\delta}(f) \mathcal{C}(T, \mu, \lambda, \bar{\lambda}) + \theta(-f) P_{(e)}(T, \mu) \right). \quad (3.3)$$

We start with the ϕ equations of motion following from the partition function (3.3)

$$\begin{aligned} \theta(f) \left[D_i \left(\frac{2}{T} \frac{\partial P_{(b)}}{\partial \chi} \zeta^i \right) \right] &= 0, \\ \tilde{\delta}(f) \left[T \tilde{D}_i \left(\frac{1}{T \bar{\lambda}} \frac{\partial \mathcal{C}}{\partial \bar{\lambda}} \tilde{\zeta}^i + \frac{1}{T} \frac{\partial \mathcal{C}}{\partial \lambda} n^i \right) + 2\lambda \frac{\partial P_{(b)}}{\partial \chi} \right] &= 0, \\ \tilde{\delta}'(f) \left[\frac{\partial \mathcal{C}}{\partial \lambda} \right] &= 0, \end{aligned} \quad (3.4)$$

where again, \tilde{D}_i denotes the covariant derivative on the surface. The energy-momentum tensor and charge current far away from the interface is exactly the same as in the 3 + 1 dimensional case. At the interface however, we can use the formulae in appendix C in order to determine the energy-momentum tensor and charge current as following (after imposing the $\tilde{\delta}'(f)$ part of the ϕ equation of motion)

$$\begin{aligned} T_{(s)00} &= e^{2\sigma} \left(-\mathcal{C} + T \frac{\partial \mathcal{C}}{\partial T} + \mu \frac{\partial \mathcal{C}}{\partial \mu} \right), \quad T_{(s)}^i{}_0 = -\xi_0 \frac{1}{\bar{\lambda}} \frac{\partial \mathcal{C}}{\partial \bar{\lambda}} \tilde{\zeta}^i, \quad T_{(s)}^{ij} = \mathcal{C} h^{ij} - \frac{1}{\bar{\lambda}} \frac{\partial \mathcal{C}}{\partial \bar{\lambda}} \tilde{\zeta}^i \tilde{\zeta}^j, \\ J_{(s)0} &= -e^\sigma \frac{\partial \mathcal{C}}{\partial \mu}, \quad J_{(s)}^i = \frac{1}{\bar{\lambda}} \frac{\partial \mathcal{C}}{\partial \bar{\lambda}} \tilde{\zeta}^i, \end{aligned} \quad (3.5)$$

where $h^{ij} = g^{ij} - n^i n^j = \epsilon^{ia} n_a \epsilon^{jb} n_b$ and $\tilde{\zeta}^i = h^{ij} \zeta_j$. Note that \mathcal{C} in (3.5) contains both parity-odd and parity-even contributions and is given by (3.2). We can also easily obtain the equation of motion for the shape-field f , which now involves parity-odd pieces as well, namely the Young-Laplace equation (after imposing the $\tilde{\delta}'(f)$ part of the ϕ equation of motion)

$$P_{(b)} - P_{(e)} + D_i \left(\frac{1}{T} \mathcal{C} n^i - \frac{\lambda}{T \bar{\lambda}} \frac{\partial \mathcal{C}}{\partial \bar{\lambda}} \tilde{\zeta}^i \right) = 0. \quad (3.6)$$

We again choose to work in a hydrodynamic frame suitable for the partition function analysis, where

$$u^\mu = e^{-\sigma} (1, 0, 0), \quad T = T_0 e^{-\sigma}, \quad \mu = e^{-\sigma} A_0, \quad (3.7)$$

²⁶Note that once we have assumed that \mathcal{C} depends both on λ and $\bar{\lambda}$, a further dependence on $\tilde{\chi}$ is redundant, since it is no longer an independent variable and is given by

$$\tilde{\chi} = \mu^2 - \bar{\lambda}^2. \quad (3.1)$$

This is due to the fact that, on the interface there are only two independent components of the superfluid velocity, the one along the surface and the one orthogonal to the surface, which are denoted by $\bar{\lambda}$ and λ respectively. We would like to emphasise, that although $\bar{\lambda}^2$ is related to $\tilde{\chi}$, there is new information in $\bar{\lambda}$ because of its parity odd nature, as compared to $\bar{\lambda}^2$, which is parity even. We should also note that, due to the relation (3.1), the surface term in (3.3) may also be written as

$$\mathcal{C}(T, \mu, \lambda, \bar{\lambda}) = \mathcal{C}_0(T, \mu, \tilde{\chi}, \lambda) + \bar{\lambda} \mathcal{C}_1(T, \mu, \tilde{\chi}, \lambda), \quad (3.2)$$

however in our discussion we choose to proceed with the form (3.3).

everywhere, including at the interface. Using the $\tilde{\delta}'(f)$ part of the equation of motion for ϕ (3.4), the covariant form of the energy-momentum tensor and current may be expressed as²⁷

$$\begin{aligned} T_{(s)}^{\mu\nu} &= (\mathcal{E} - \mathcal{Y})u^\mu u^\nu - \mathcal{Y}(\mathcal{G}^{\mu\nu} - n^\mu n^\nu) + \mathcal{F}\tilde{\xi}^\mu \tilde{\xi}^\nu, \\ J_{(s)}^\mu &= \mathcal{Q}u^\mu - \mathcal{F}\tilde{\xi}^\mu, \end{aligned} \quad (3.10)$$

according to which comparison with (3.5) allows us to obtain

$$\begin{aligned} \mathcal{E} &= -\mathcal{C} + T \frac{\partial \mathcal{C}}{\partial T} + \mu \frac{\partial \mathcal{C}}{\partial \mu} + \frac{\mu^2}{\bar{\lambda}} \frac{\partial \mathcal{C}}{\partial \bar{\lambda}}, & \mathcal{Y} &= -\mathcal{C}, & \mathcal{F} &= -\frac{1}{\bar{\lambda}} \frac{\partial \mathcal{C}}{\partial \bar{\lambda}}, \\ \mathcal{Q} &= \frac{\partial \mathcal{C}}{\partial \mu} + \frac{\mu}{\bar{\lambda}} \frac{\partial \mathcal{C}}{\partial \bar{\lambda}}, & \mathcal{S} &= \frac{\partial \mathcal{C}}{\partial T}. \end{aligned} \quad (3.11)$$

Here $\partial \mathcal{C} / \partial \bar{\lambda}$ is zero on-shell due to the $\tilde{\delta}'(f)$ equation of motion for ϕ (3.4). \mathcal{S} is again the surface entropy current, entering the entropy current as $J_{(s)\text{ent}}^\mu = \mathcal{S}u^\mu$. These relations can be summarized as the surface Euler relation and the first law of thermodynamics specific to 2 + 1 dimensions, respectively as

$$\mathcal{E} - \mathcal{Y} = T\mathcal{S} + \mu\mathcal{Q}, \quad d\mathcal{Y} = -\mathcal{S}dT - \mathcal{Q}d\mu - \mathcal{F}(\mu d\mu - \bar{\lambda}d\bar{\lambda}). \quad (3.12)$$

Note that, if we do not have any parity-odd dependence in the surface tension \mathcal{Y} , i.e. if it only depends on $\bar{\lambda}^2$, the final differential becomes $\mu d\mu - \bar{\lambda}d\bar{\lambda} = \frac{1}{2}d(\mu^2 - \bar{\lambda}^2) = \frac{1}{2}d\tilde{\chi}$ and we get the familiar perfect superfluid surface first law of thermodynamics as in (2.8). The first law of thermodynamics in this case however is slightly subtle, and we have discussed it in appendix A.

3.2 First order corrections away from the interface

We now wish to extend the results of the previous section by considering first derivative corrections to the partition function of the bulk superfluid and that of the exterior charged fluid, which by the same “inflow” mechanism explained in section 2.2, will affect the surface currents in an important way. A significant difference between 2 + 1 dimensional superfluid bubbles and that of its 3 + 1 dimensional counterpart, is the fact that the partition function of the exterior charged fluid also receives non-zero contributions at first order in derivatives.

²⁷In 2 + 1 dimensions there can at most be 3 independent vectors but we have at least 6 on the surface: u^μ , $\tilde{\xi}^\mu$, n^μ , $\epsilon^{\mu\nu\rho}u_\nu n_\rho$, $\epsilon^{\mu\nu\rho}u_\nu \tilde{\xi}_\rho$, $\epsilon^{\mu\nu\rho}n_\nu \tilde{\xi}_\rho$. Choosing a basis of any three, we can write the others in terms of the chosen basis, for example choosing u^μ , $\tilde{\xi}^\mu$, n^μ (as we did in (3.10))

$$\epsilon^{\mu\nu\rho}u_\nu n_\rho = -\frac{1}{\bar{\lambda}}(\tilde{\xi}^\mu + \mu u^\mu), \quad \epsilon^{\mu\nu\rho}u_\nu \tilde{\xi}_\rho = \bar{\lambda}n^\mu, \quad \epsilon^{\mu\nu\rho}n_\nu \tilde{\xi}_\rho = -\frac{\mu}{\bar{\lambda}}\tilde{\xi}^\mu - \frac{\mu^2 - \bar{\lambda}^2}{\bar{\lambda}}u^\mu. \quad (3.8)$$

This allows us to write the constitutive relations (3.10) in many other basis. For example, choosing $\epsilon^{\mu\nu\rho}u_\nu n_\rho$ in favour of $\tilde{\xi}^\mu$ we find

$$\begin{aligned} T_{(s)}^{\mu\nu} &= (\mathcal{E}' - \mathcal{Y}')u^\mu u^\nu - \mathcal{Y}'(\mathcal{G}^{\mu\nu} - n^\mu n^\nu) + 2\mathcal{F}'\mu u^{(\mu} \epsilon^{\nu)\sigma\rho}u_\sigma n_\rho, \\ J_{(s)}^\mu &= \mathcal{Q}'u^\mu + \mathcal{F}'\epsilon^{\mu\nu\rho}u_\nu n_\rho, \end{aligned} \quad (3.9)$$

where $\mathcal{E}' = \mathcal{E} + \mu^2 \mathcal{F}$, $\mathcal{Y}' = \mathcal{Y} - \bar{\lambda}^2 \mathcal{F}$, $\mathcal{Q}' = \mathcal{Q} + \mu \mathcal{F}$ and $\mathcal{F}' = \bar{\lambda} \mathcal{F}$ in terms of the coefficients in (3.10).

The total first order partition function $W^{(1)}$ for 2+1 dimensional superfluid bubbles can be expressed in terms of the parity-odd corrections to the bulk superfluid partition function $W_{\text{odd}}^{(1)}$ and the parity-odd corrections to the exterior charged fluid partition function $\tilde{W}_{\text{odd}}^{(1)}$ such that

$$W^{(1)} = W_{\text{even}}^{(1)} + W_{\text{odd}}^{(1)} + \tilde{W}_{\text{odd}}^{(1)}, \quad (3.13)$$

where $W_{\text{even}}^{(1)}$ corresponds to the first order corrections to the partition function of the bulk superfluid in the parity-even sector (2.11), since these corrections are universal irrespective of spacetime dimensions.

The parity-odd first order corrections in the bulk of the 2 + 1 dimensional superfluid are significantly different from the 3 + 1 dimensional case. They are given by

$$W_{\text{odd}}^{(1)} = \int d^2x \sqrt{g} \frac{e^\sigma}{T_0} \theta(f) (m_\omega \epsilon^{ij} \partial_i a_j + m_B \epsilon^{ij} \partial_i A_j + \beta_1 \epsilon^{ij} \zeta_i \partial_j \sigma + \beta_2 \epsilon^{ij} \zeta_i \partial_j A_0). \quad (3.14)$$

On the other hand, the parity-odd corrections to the partition function of the exterior charged fluid read [11]

$$\tilde{W}_{\text{odd}}^{(1)} = \int d^2x \sqrt{g} \frac{e^\sigma}{T_0} \theta(f) (M_\omega \epsilon^{ij} \partial_i a_j + M_B \epsilon^{ij} \partial_i A_j). \quad (3.15)$$

The coefficients $m_\omega, m_B, \beta_1, \beta_2$ in (3.14) depend on the three scalars T, μ and χ , while the coefficients M_ω, M_B parametrizing the charged fluid in (3.15) only depend on T and μ .

The bulk currents that follow from direct variation of (3.15) were studied in [11], while the surface effects were recently considered in [9]. Therefore, we defer the reader to these references for more details on these currents. However, neither the bulk nor the surface effects of (3.14) have been previously analyzed in the literature. Below, we explicitly provide the bulk and surface currents that follow from (3.14). In section 3.2.1, we obtain the constraints among bulk transport coefficients that $W_{\text{odd}}^{(1)}$ imposes on the 2 + 1 dimensional superfluid, while in section 3.2.2 we obtain the constraints imposed by it on the surface transport coefficients. Finally, in section 3.2.3 we study the rich thermodynamic properties of the interface between the bulk superfluid and the exterior charged fluid by considering all surface effects arising from $W^{(1)}$.

Bulk currents. The bulk energy-momentum tensor and the charge current obtained by varying (3.14) using the formulae of appendix C take the form

$$\begin{aligned} T_{(b)00} = & -e^{2\sigma} \left[\left(m_\omega - T \frac{\partial m_\omega}{\partial T} - 2e^{-2\sigma} \xi_0^2 \frac{\partial m_\omega}{\partial \chi} \right) \epsilon^{ij} \partial_i a_j \right. \\ & + \left(m_B - T \frac{\partial m_B}{\partial T} - 2e^{-2\sigma} \xi_0^2 \frac{\partial m_B}{\partial \chi} + \beta_1 \right) \epsilon^{ij} \partial_i A_j \\ & + \left(\beta_2 - T \frac{\partial \beta_2}{\partial T} - 2e^{-2\sigma} \xi_0^2 \frac{\partial \beta_2}{\partial \chi} - \frac{1}{T_0} \frac{\partial \beta_1}{\partial \nu} \right) \epsilon^{ij} \zeta_i \partial_j A_0 \\ & \left. - 2e^{-2\sigma} \xi_0^2 \frac{\partial \beta_1}{\partial T} \epsilon^{ij} \zeta_i \partial_j \sigma - \frac{\partial \beta_1}{\partial \chi} \epsilon^{ij} \zeta_i \partial_j \chi \right], \end{aligned} \quad (3.16)$$

$$\begin{aligned}
 T_{(b)}^i{}_0 &= \left(m_\omega - A_0(\beta_1 + m_B) - T \left(\frac{\partial m_\omega}{\partial T} - A_0 \frac{\partial m_B}{\partial T} \right) \right) \epsilon^{ij} \partial_j \sigma \\
 &+ \left(\frac{1}{T_0} \left(\frac{\partial m_\omega}{\partial \nu} - A_0 \frac{\partial m_B}{\partial \nu} \right) - A_0 \beta_2 \right) \epsilon^{ij} \partial_j A_0 + \left(\frac{\partial m_\omega}{\partial \chi} - A_0 \frac{\partial m_B}{\partial \chi} \right) \epsilon^{ij} \partial_j \chi \\
 &+ 2A_0 \xi^i \left(\frac{\partial m_\omega}{\partial \chi} \epsilon^{jk} \partial_j a_k + \frac{\partial m_B}{\partial \chi} \epsilon^{jk} \partial_j A_k + \frac{\partial \beta_1}{\partial \chi} \epsilon^{jk} \zeta_j \partial_k \sigma + \frac{\partial \beta_2}{\partial \chi} \epsilon^{jk} \zeta_j \partial_k A_0 \right), \quad (3.17)
 \end{aligned}$$

$$T_{(b)}^{ij} = 2\xi^i \xi^j \left[\frac{\partial m_\omega}{\partial \chi} \epsilon^{kl} \partial_k a_l + \frac{\partial m_B}{\partial \chi} \epsilon^{kl} \partial_k A_l + \frac{\partial \beta_1}{\partial \chi} \epsilon^{kl} \zeta_k \partial_l \sigma + \frac{\partial \beta_2}{\partial \chi} \epsilon^{kl} \zeta_k \partial_l A_0 \right], \quad (3.18)$$

$$\begin{aligned}
 J_{(b)0} &= -e^{2\sigma} \left[\left(\frac{1}{T_0} \frac{\partial m_\omega}{\partial \nu} + 2A_0 e^{-2\sigma} \frac{\partial m_\omega}{\partial \chi} \right) \epsilon^{ij} \partial_i a_j \right. \\
 &+ \left(\frac{1}{T_0} \frac{\partial m_B}{\partial \nu} + 2A_0 e^{-2\sigma} \frac{\partial m_B}{\partial \chi} + \beta_2 \right) \epsilon^{ij} \partial_i A_j \\
 &+ \left(\frac{1}{T_0} \frac{\partial \beta_1}{\partial \nu} + 2A_0 e^{-2\sigma} \frac{\partial \beta_1}{\partial \chi} - \beta_2 + T \frac{\partial \beta_2}{\partial T} \right) \epsilon^{ij} \zeta_i \partial_j \sigma \\
 &\left. + 2A_0 e^{-2\sigma} \frac{\partial \beta_2}{\partial \chi} \epsilon^{ij} \zeta_i \partial_j A_0 - \frac{\partial \beta_2}{\partial \chi} \epsilon^{ij} \zeta_i \partial_j \chi \right], \quad (3.19)
 \end{aligned}$$

$$\begin{aligned}
 J_{(b)}^i &= -2\xi^i \left(\frac{\partial m_\omega}{\partial \chi} \epsilon^{jk} \partial_j a_k + \frac{\partial m_B}{\partial \chi} \epsilon^{jk} \partial_j A_k + \frac{\partial \beta_1}{\partial \chi} \epsilon^{jk} \zeta_j \partial_k \sigma + \frac{\partial \beta_2}{\partial \chi} \epsilon^{jk} \zeta_j \partial_k A_0 \right) \\
 &+ \left(\beta_1 + m_B - T \frac{\partial m_B}{\partial T} \right) \epsilon^{ij} \partial_j \sigma + \left(\beta_2 + \frac{1}{T_0} \frac{\partial m_B}{\partial \nu} \right) \epsilon^{ij} \partial_j A_0 + \frac{\partial m_B}{\partial \chi} \epsilon^{ij} \partial_j \chi. \quad (3.20)
 \end{aligned}$$

It is important to note that the bulk energy-momentum tensor and current are entirely determined in terms of the coefficients $m_\omega, m_B, \beta_1, \beta_2$.

Surface currents. The surface energy-momentum tensor and charge current obtained by varying (3.14) take the form

$$\begin{aligned}
 T_{(s)}^{ij} &= 0, & T_{(s)00} &= e^{2\sigma} \bar{\lambda} \beta_1, & T_{(s)0}^i &= -(m_\omega - A_0 m_B) \epsilon^{ij} n_j, \\
 J_{(s)0} &= \bar{\lambda} \beta_2 e^{2\sigma}, & J_{(s)}^i &= -m_B \epsilon^{ij} n_j.
 \end{aligned} \quad (3.21)$$

The effect of the coefficients m_ω and m_B appearing in (3.14) on the surface currents is essentially the same as the effect of the coefficients M_ω and M_B appearing in (3.15). In fact, the contributions to the surface currents due to the parity-odd sector of $W^{(1)}$ is entirely given by (3.21) with the replacement $m_\omega \rightarrow m_\omega - M_\omega$ and $m_B \rightarrow m_B - M_B$.

3.2.1 Constraints on the bulk parity-odd constitutive relations

In this section we derive the constraints on the covariant form of the bulk energy-momentum tensor and charge current that are implied by eqs. (3.16)–(3.20), which in turn follow from the partition function (3.14). In order to do so, one must classify all first order parity-odd terms, which are non-zero in equilibrium, based on their transformation properties under the spatial rotation group. The possible scalars have been listed in table 1, while the vectors orthogonal to both u^μ and ξ^μ have been listed in table 2.²⁸

²⁸Note that in 2+1 dimensions it is not possible to write any second rank symmetric tensor invariant under the spatial rotation group with both indices projected orthogonal to both u^μ and ξ^μ .

| a | $\mathcal{S}_{(a)}$ | Reduced form $\mathcal{S}_{(a)}^{\text{red}}$ |
|-----|---|---|
| 1 | $\epsilon^{\sigma\nu\lambda} u_\sigma \omega_{\nu\lambda}$ | $e^\sigma \epsilon^{ij} \partial_i a_j$ |
| 2 | $\epsilon^{\sigma\nu\lambda} u_\sigma \mathcal{F}_{\nu\lambda}$ | $-2\epsilon^{ij} (\partial_i A_j + A_0 \partial_i a_j)$ |
| 3 | $\epsilon^{\sigma\nu\lambda} \xi_\sigma u_\nu \partial_\lambda T$ | $\epsilon^{ij} \zeta_i \partial_j T = -T \epsilon^{ij} \zeta_i \partial_j \sigma$ |
| 4 | $\epsilon^{\sigma\nu\lambda} \xi_\sigma u_\nu \partial_\lambda \nu$ | $\epsilon^{ij} \zeta_i \partial_j \nu = \frac{1}{T_0} \epsilon^{ij} \zeta_i \partial_j A_0$ |
| 5 | $-\epsilon^{\sigma\nu\lambda} \xi_\sigma u_\nu \partial_\lambda \chi$ | $-\epsilon^{ij} \zeta_i \partial_j \chi$ |

Table 1. Parity-odd first order scalars in 2 + 1 dimensions and their dimensional reduction.

| a | $\mathcal{V}_{(a)}^\mu$ | Reduced form $\mathcal{V}_{(a)}^i$ |
|-----|---|---|
| 1 | $\tilde{P}_\sigma^\mu \epsilon^{\sigma\nu\lambda} u_\nu \partial_\lambda T$ | $\tilde{P}^{ik} g_{kl} \epsilon^{lj} \partial_j T = -T \tilde{P}^{ik} g_{kl} \epsilon^{lj} \partial_j \sigma$ |
| 2 | $\tilde{P}_\sigma^\mu \epsilon^{\sigma\nu\lambda} u_\nu \partial_\lambda \nu$ | $\tilde{P}^{ik} g_{kl} \epsilon^{lj} \partial_j \nu = \frac{1}{T_0} \tilde{P}^{ik} g_{kl} \epsilon^{lj} \partial_j A_0$ |
| 3 | $-\tilde{P}_\sigma^\mu \epsilon^{\sigma\nu\lambda} u_\nu \partial_\lambda \chi$ | $-\tilde{P}^{ik} g_{kl} \epsilon^{lj} \partial_j \chi$ |

Table 2. Parity-odd vectors in 2 + 1 dimensions and their dimensional reduction. Here we have defined the projector $\tilde{P}_{\mu\nu} = \mathcal{G}_{\mu\nu} + u_\mu u_\nu - \frac{\zeta^\mu \zeta^\nu}{|\zeta|^2}$, where $\zeta^\mu = \xi^\mu + (u^\nu \xi_\nu) u^\mu$. After the reduction on the time circle, we find $(\mathcal{V}_{(a)})_0 = 0$ while $\mathcal{V}_{(a)}^i$ is specified in the right column. Also, after the reduction, the tangential projector takes the form $\tilde{P}^{ij} = g^{ij} - \frac{\zeta^i \zeta^j}{(g^{ik} \zeta_i \zeta_k)}$, $\tilde{P}_{00} = P^i{}_0 = 0$.

Using tables 1 and 2, the most general bulk energy-momentum tensor and charge current allowed by symmetries, describing 2 + 1 superfluid bubbles in the parity-odd sector at first order in derivatives are

$$\begin{aligned}
 T_{(b)}^{\mu\nu} &= \sum_{a=1}^5 \left(\mathfrak{s}_a^{(1)} \mathcal{S}_{(a)} u^\mu u^\nu + \mathfrak{s}_a^{(2)} \mathcal{S}_{(a)} \mathcal{G}^{\mu\nu} + \mathfrak{s}_a^{(3)} \mathcal{S}_{(a)} \zeta^\mu \zeta^\nu + \mathfrak{s}_a^{(4)} \mathcal{S}_{(a)} \zeta^{(\mu} u^{\nu)} \right) \\
 &\quad + \sum_{a=1}^3 \left(\mathfrak{v}_a^{(1)} u^{(\mu} \mathcal{V}_{(a)}^{\nu)} + \mathfrak{v}_a^{(2)} \zeta^{(\mu} \mathcal{V}_{(a)}^{\nu)} \right), \\
 J_{(b)}^\mu &= \sum_{a=1}^5 \left(\mathfrak{s}_a^{(5)} \mathcal{S}_{(a)} u^\mu + \mathfrak{s}_a^{(6)} \mathcal{S}_{(a)} \zeta^\mu \right) + \sum_{a=1}^3 \mathfrak{v}_a^{(3)} \mathcal{V}_{(a)}^\mu,
 \end{aligned} \tag{3.22}$$

where $\zeta^\mu = \xi^\mu + (u^\nu \xi_\nu) u^\mu$ is the superfluid velocity projected orthogonally to the fluid velocity. The energy-momentum tensor and charge current in (3.23) are parametrized by a total of 39 transport coefficients, which will ultimately be constrained in terms of the four parameters $m_\omega, m_B, \beta_1, \beta_2$ appearing in the partition function (3.14). Using the explicit reductions provided in tables 1 and 2, we can readily reduce the energy-momentum tensor

and current given in (3.22) into

$$\begin{aligned}
 T_{(b)00} &= e^{2\sigma} \sum_{a=1}^5 \mathcal{S}_{(a)}^{\text{red}} (\mathfrak{s}_a^{(1)} - \mathfrak{s}_a^{(2)}), & T_{(b)0}^i &= -\frac{e^\sigma}{2} \left[\sum_{a=1}^5 \mathcal{S}_{(a)}^{\text{red}} \mathfrak{s}_a^{(4)} \zeta^i + \sum_{a=1}^3 \mathfrak{v}_a^{(1)} \mathcal{V}_{(a)}^i \right], \\
 T_{(b)}^{ij} &= \sum_{a=1}^5 \mathcal{S}_{(a)}^{\text{red}} \left\{ \mathfrak{s}_a^{(2)} g^{ij} - \mathfrak{s}_a^{(3)} \zeta^i \zeta^j \right\} + \sum_{a=1}^3 \mathfrak{v}_a^{(2)} \zeta^i \mathcal{V}_{(a)}^j, \\
 J_{(b)0} &= -e^\sigma \sum_{a=1}^5 \mathfrak{s}_a^{(5)} \mathcal{S}_{(a)}^{\text{red}}, & J_{(b)}^i &= \sum_{a=1}^5 \mathfrak{s}_a^{(6)} \zeta^i \mathcal{S}_{(a)}^{\text{red}} + \sum_{a=1}^3 \mathfrak{v}_a^{(3)} \mathcal{V}_{(a)}^i.
 \end{aligned} \tag{3.23}$$

Comparing the components of the energy-momentum tensor and current in eq. (3.23) with the corresponding expressions in (3.16)–(3.20) that follow from the partition function, leads to the following four sets of relations among transport coefficients

$$\begin{aligned}
 \mathfrak{s}_a^{(2)} &= 0, \quad (a = 1, 2, 3, 4, 5), \quad \mathfrak{v}_a^{(2)} = 0, \quad (a = 1, 2, 3), \\
 \mathfrak{v}_3^{(3)} &= -\frac{\partial m_B}{\partial \chi}, \quad \mathfrak{s}_2^{(6)} = \frac{\partial m_B}{\partial \chi}, \quad \mathfrak{s}_2^{(3)} = \frac{\partial m_B}{\partial \chi}, \quad \mathfrak{s}_2^{(4)} = 2\nu T \frac{\partial m_B}{\partial \chi}, \quad \mathfrak{s}_5^{(6)} = -\frac{\mathfrak{v}_3^{(3)}}{|\zeta|^2}, \\
 \mathfrak{s}_5^{(3)} &= \frac{\mathfrak{v}_2^{(3)}}{|\zeta|^2}, \quad \mathfrak{s}_5^{(4)} = \frac{\mathfrak{v}_3^{(1)}}{|\zeta|^2}, \quad \mathfrak{s}_1^{(6)} = -2e^{-\sigma} \left(\frac{\partial m_\omega}{\partial \chi} - A_0 \frac{\partial m_B}{\partial \chi} \right), \\
 \mathfrak{s}_1^{(3)} &= -2e^{-\sigma} \left(\frac{\partial m_\omega}{\partial \chi} - A_0 \frac{\partial m_B}{\partial \chi} \right), \quad \mathfrak{v}_3^{(1)} = 2e^{-\sigma} \left(\frac{\partial m_\omega}{\partial \chi} - A_0 \frac{\partial m_B}{\partial \chi} \right), \\
 \mathfrak{s}_1^{(4)} &= 4\nu T e^{-\sigma} \left(\frac{\partial m_\omega}{\partial \chi} - A_0 \frac{\partial m_B}{\partial \chi} \right), \quad \mathfrak{s}_5^{(1)} = \frac{\partial \beta_1}{\partial \chi}, \quad \mathfrak{s}_3^{(1)} = -2T\nu^2 \frac{\partial \beta_1}{\partial \chi},
 \end{aligned} \tag{3.24}$$

$$\begin{aligned}
 \mathfrak{s}_3^{(3)} &= \frac{2}{T} \frac{\partial \beta_1}{\partial \chi}, \quad \mathfrak{s}_3^{(4)} - \frac{\mathfrak{v}_1^{(1)}}{|\zeta|^2} = 4\nu \frac{\partial \beta_1}{\partial \chi}, \quad \mathfrak{s}_3^{(6)} + \frac{\mathfrak{v}_1^{(3)}}{|\zeta|^2} = \frac{2}{T} \frac{\partial \beta_1}{\partial \chi}, \\
 \mathfrak{s}_4^{(5)} &= 2\nu T \frac{\partial(T_0 \beta_2)}{\partial \chi}, \quad \mathfrak{s}_5^{(5)} = \frac{1}{T} \frac{\partial(T_0 \beta_2)}{\partial \chi}, \quad \mathfrak{s}_4^{(3)} = -2 \frac{\partial(T_0 \beta_2)}{\partial \chi}, \\
 \mathfrak{s}_4^{(6)} + \frac{\mathfrak{v}_2^{(3)}}{|\zeta|^2} &= -2 \frac{\partial(T_0 \beta_2)}{\partial \chi}, \quad \mathfrak{s}_4^{(4)} + \frac{\mathfrak{v}_2^{(1)}}{|\zeta|^2} = 4\nu T \frac{\partial(T_0 \beta_2)}{\partial \chi}, \quad \mathfrak{v}_2^{(3)} = T_0 \beta_2 + \frac{\partial m_B}{\partial \nu},
 \end{aligned} \tag{3.25}$$

$$\begin{aligned}
 \mathfrak{s}_2^{(5)} &= \frac{1}{2T} \left[T_0 \beta_2 + \frac{\partial m_B}{\partial \nu} + 2\nu T^2 \frac{\partial m_B}{\partial \chi} \right], \quad \mathfrak{v}_1^{(3)} = -\frac{1}{T} \left(\beta_1 + m_B - T \frac{\partial m_B}{\partial T} \right), \\
 \mathfrak{s}_2^{(1)} &= \frac{1}{2} \left[\beta_1 + m_B - T \frac{\partial m_B}{\partial T} - 2\nu^2 T^2 \frac{\partial m_B}{\partial \chi} \right], \\
 \mathfrak{s}_1^{(5)} &= -\frac{e^{-\sigma}}{T} \left[\left(\frac{\partial m_\omega}{\partial \nu} - A_0 \frac{\partial m_B}{\partial \nu} \right) + 2\nu T^2 \left(\frac{\partial m_\omega}{\partial \chi} - A_0 \frac{\partial m_B}{\partial \chi} \right) \right] - \nu T_0 \beta_2, \\
 \mathfrak{s}_1^{(1)} &= \nu \beta_1 - e^{-\sigma} \left[(m_\omega - A_0 m_B) - T \left(\frac{\partial m_\omega}{\partial T} - A_0 \frac{\partial m_B}{\partial T} \right) \right. \\
 &\quad \left. - 2\nu^2 T^2 \left(\frac{\partial m_\omega}{\partial \chi} - A_0 \frac{\partial m_B}{\partial \chi} \right) \right],
 \end{aligned} \tag{3.26}$$

$$\begin{aligned}
 \mathfrak{s}_3^{(5)} &= -\frac{1}{T^2} \left[\frac{\partial \beta_1}{\partial \nu} - T_0 \beta_2 + T \frac{\partial(T_0 \beta_2)}{\partial T} + 2\nu T^2 \frac{\partial \beta_1}{\partial \chi} \right], \\
 \mathfrak{s}_4^{(1)} &= \frac{\partial \beta_1}{\partial \nu} - T_0 \beta_2 + T \frac{\partial(T_0 \beta_2)}{\partial T} + 2\nu^2 T^2 \frac{\partial(T_0 \beta_2)}{\partial \chi},
 \end{aligned}$$

$$\begin{aligned}
 \mathfrak{v}_1^{(1)} &= \frac{2}{T} \left[(m_\omega - A_0 m_B) - T \left(\frac{\partial m_\omega}{\partial T} - A_0 \frac{\partial m_B}{\partial T} \right) \right] - 2\nu\beta_1, \\
 \mathfrak{v}_2^{(1)} &= 2\nu T(T_0\beta_2) - 2e^{-\sigma} \left(\frac{\partial m_\omega}{\partial \nu} - A_0 \frac{\partial m_B}{\partial \nu} \right).
 \end{aligned} \tag{3.27}$$

Eqs. (3.24)–(3.27) provide a total of 39 relations among the transport coefficients. We will now eliminate the coefficients m_ω , m_B , β_1 , β_2 from the relations in eqs. (3.24)–(3.27) in order to obtain 35 independent relations among the remaining 35 transport coefficients present in eq. (3.22). These relations are listed below

$$\begin{aligned}
 \mathfrak{s}_a^{(2)} &= 0 \quad (a = 1, 2, 3, 4, 5), \quad \mathfrak{v}_a^{(2)} = 0 \quad (a = 1, 2, 3), \quad \mathfrak{v}_3^{(3)} = -\mathfrak{s}_2^{(6)}, \quad \mathfrak{s}_2^{(3)} = \mathfrak{s}_2^{(6)}, \\
 \mathfrak{s}_2^{(4)} &= 2\nu T \mathfrak{s}_2^{(6)}, \quad \mathfrak{s}_5^{(6)} = -\frac{\mathfrak{v}_3^{(3)}}{|\zeta|^2}, \quad \mathfrak{s}_5^{(3)} = \frac{\mathfrak{v}_2^{(3)}}{|\zeta|^2}, \quad \mathfrak{s}_5^{(4)} = \frac{\mathfrak{v}_3^{(1)}}{|\zeta|^2}, \quad \mathfrak{s}_1^{(3)} = \mathfrak{s}_1^{(6)}, \\
 \mathfrak{v}_3^{(1)} &= \mathfrak{s}_1^{(6)}, \quad \mathfrak{s}_1^{(4)} = -2\nu T \mathfrak{s}_1^{(6)}, \quad \mathfrak{s}_3^{(1)} = -2T\nu^2 \mathfrak{s}_5^{(1)}, \quad \mathfrak{s}_3^{(3)} = \frac{2}{T} \mathfrak{s}_5^{(1)}, \\
 \mathfrak{s}_3^{(4)} - \frac{\mathfrak{v}_1^{(1)}}{|\zeta|^2} &= 4\nu \mathfrak{s}_5^{(1)}, \quad \mathfrak{s}_3^{(6)} + \frac{\mathfrak{v}_1^{(3)}}{|\zeta|^2} = \frac{2}{T} \mathfrak{s}_5^{(1)}, \quad \mathfrak{s}_4^{(5)} = 2\nu T^2 \mathfrak{s}_5^{(5)}, \quad \mathfrak{s}_4^{(3)} = -2T \mathfrak{s}_5^{(5)},
 \end{aligned} \tag{3.28}$$

$$\begin{aligned}
 \mathfrak{s}_4^{(6)} + \frac{\mathfrak{v}_2^{(3)}}{|\zeta|^2} &= -2T \mathfrak{s}_5^{(5)}, \quad \mathfrak{s}_4^{(4)} + \frac{\mathfrak{v}_2^{(1)}}{|\zeta|^2} = 4\nu T^2 \mathfrak{s}_5^{(5)}, \quad \mathfrak{s}_2^{(5)} = \frac{1}{2T} \left[\mathfrak{v}_2^{(3)} + 2\nu T^2 \mathfrak{s}_2^{(6)} \right], \\
 \mathfrak{s}_2^{(1)} &= -\frac{T}{2} \left[\mathfrak{v}_1^{(3)} + 2\nu T \mathfrak{s}_2^{(6)} \right], \quad \mathfrak{s}_1^{(1)} = -\frac{T}{2} \mathfrak{v}_1^{(1)} + \nu^2 T^2 \mathfrak{s}_1^{(6)}, \quad \mathfrak{s}_1^{(5)} = -\frac{\mathfrak{v}_2^{(1)}}{2T} + \nu T \mathfrak{s}_1^{(6)}, \\
 \mathfrak{s}_4^{(1)} &= T^2 \left[2\nu^2 T \mathfrak{s}_5^{(5)} - \mathfrak{s}_3^{(5)} - 2\nu \mathfrak{s}_5^{(1)} \right], \quad \mathfrak{s}_5^{(5)} = \frac{1}{T} \left(\frac{\partial \mathfrak{v}_2^{(3)}}{\partial \chi} - \frac{\partial \mathfrak{s}_2^{(6)}}{\partial \nu} \right),
 \end{aligned} \tag{3.29}$$

$$\begin{aligned}
 \mathfrak{v}_2^{(3)} &= T \frac{\partial \mathfrak{v}_2^{(3)}}{\partial T} + T^2 \mathfrak{s}_3^{(5)} - T \frac{\partial \mathfrak{v}_1^{(3)}}{\partial \nu} + 2\nu T^2 \mathfrak{s}_5^{(1)}, \\
 \mathfrak{s}_5^{(1)} &= - \left[\mathfrak{s}_2^{(6)} - T \frac{\partial \mathfrak{s}_2^{(6)}}{\partial T} + T \frac{\partial \mathfrak{v}_1^{(3)}}{\partial \chi} \right], \quad \mathfrak{s}_2^{(6)} = \frac{1}{2T} \left[\frac{\partial \mathfrak{s}_1^{(6)}}{\partial \nu} - \frac{\partial \mathfrak{v}_2^{(1)}}{\partial \chi} \right] - \nu T \mathfrak{s}_5^{(5)}, \\
 \mathfrak{s}_1^{(6)} &= -\frac{T}{2} \left[\frac{\partial \mathfrak{v}_1^{(1)}}{\partial \chi} + 2\nu \mathfrak{s}_5^{(1)} - \frac{\partial \mathfrak{s}_1^{(6)}}{\partial T} \right].
 \end{aligned} \tag{3.30}$$

As expected, the remaining 35 transport coefficients are determined in terms of the four coefficients $m_\omega, m_B, \beta_1, \beta_2$.

3.2.2 Bulk parity-odd effects on the surface currents

Following the same strategy as in the previous section, here we derive constraints on the covariant form of the surface energy-momentum tensor and charge current. These constraints are implied by (3.21), which in turn follows from the partition function (3.14).

The only parity-odd scalar we can write at the surface is actually $\bar{\lambda} = \epsilon^{\mu\nu\rho} n_\mu u_\nu \xi_\rho$, which upon dimensional reduction is equal to $\bar{\lambda} = \epsilon^{ij} n_i \zeta_j$. On the other hand, there are no new independent parity-odd vectors or tensors. The reason being that at the surface we are supposed to write tensor structures transverse to all three of u^μ , ξ^μ and n^μ , and in $2+1$ dimensions, there are no such possible tensors structures. Another way to see this is that in

2 + 1 dimensions, any vector or tensor can be expressed in terms of a chosen basis of three vectors, which we naturally have at the surface as u^μ , ξ^μ and n^μ . Therefore, there can be no other vectors or tensors in 2 + 1 dimensions, which are not their linear combinations. For example, $\mathcal{G}^{\mu\nu} = -u^\mu u^\nu + \frac{1}{|\xi|^2} \tilde{\zeta}^\mu \tilde{\zeta}^\nu + n^\mu n^\nu$, where $\tilde{\zeta}^\mu = \xi^\mu + (u^\nu \xi_\nu) u^\mu - (n^\nu \xi_\nu) n^\mu$. Having said that, in this sector (the sector of parity-odd transport flowing in from the bulk) we work with an alternate basis, u^μ , n^μ and $\bar{n}^\mu = \epsilon^{\mu\nu\rho} u_\nu n_\rho$, i.e. exchanging ξ^μ for \bar{n}^μ . This basis is more appropriate because it is simultaneously valid for the “inflow” from the exterior ordinary fluid, where $\xi^\mu = 0$. Note that now, $\mathcal{G}^{\mu\nu} = -u^\mu u^\nu + \bar{n}^\mu \bar{n}^\nu + n^\mu n^\nu$.

Following our discussion above, the most general parity-odd ideal order surface energy-momentum tensor and charge current allowed by symmetries for 2+1 dimensional superfluid bubbles is given as

$$\begin{aligned} T_{(s)}^{\mu\nu} &= \left[\mathfrak{a}_1 u^\mu u^\nu + \mathfrak{a}_2 \bar{n}^\mu \bar{n}^\nu + \mathfrak{a}_3 n^\mu n^\nu + 2\mathfrak{a}_4 u^{(\mu} n^{\nu)} \right] \bar{\lambda} + 2\mathfrak{b}_1 u^{(\mu} \bar{n}^{\nu)} + 2\mathfrak{b}_2 n^{(\mu} \bar{n}^{\nu)}, \\ J_{(s)}^\mu &= \left[\mathfrak{a}_5 u^\mu + \mathfrak{a}_6 n^\mu \right] \bar{\lambda} + \mathfrak{b}_3 \bar{n}^\mu, \end{aligned} \quad (3.31)$$

where all the transport coefficients are parity-even, i.e. they do not have $\bar{\lambda}$ dependence. The energy-momentum tensor and charge current (3.31) are parametrized by a total of 9 transport coefficients. As we will see below, the partition function (3.14) will give 5 relations among these 9 coefficients, while determining the other 4 in terms of the four parameters appearing in (3.14). It is straightforward to compare the surface energy-momentum tensor and charge current (3.31) with that in (3.21) that follow from the partition function. This comparison leads to the following relations 9 relations

$$\begin{aligned} \mathfrak{a}_1 &= \beta_1, \quad \mathfrak{a}_5 = -\frac{T_0}{T} \beta_2, \quad \mathfrak{b}_1 = \frac{T}{T_0} (m_\omega - A_0 m_B), \quad \mathfrak{b}_3 = -m_B, \\ \mathfrak{a}_2 &= \mathfrak{a}_3 = \mathfrak{a}_4 = \mathfrak{a}_6 = \mathfrak{b}_2 = 0. \end{aligned} \quad (3.32)$$

As expected, all surface transport coefficients are determined in terms of the four coefficients m_ω , m_B , β_1 and β_2 that appear in the partition function (3.14).

Finally, before concluding this section, we would like to point out that instead of using the partition function (3.31) written in the reduced language in two dimensions, we could have used its covariant version in 2 + 1 dimensions, which takes the form

$$\begin{aligned} W_{odd}^{(1)} &= \int d^3x \sqrt{-\mathcal{G}} \theta(f) \left(\kappa_1 \epsilon^{\mu\nu\lambda} u_\mu \omega_{\nu\lambda} + \frac{1}{2} \kappa_2 \epsilon^{\mu\nu\lambda} u_\mu \mathcal{F}_{\nu\lambda} \right. \\ &\quad \left. + \kappa_3 \epsilon^{\mu\nu\lambda} \zeta_\mu u_\nu \frac{1}{T} \partial_\lambda T + \kappa_4 \epsilon^{\mu\nu\lambda} \zeta_\mu u_\nu T \partial_\lambda \nu \right), \end{aligned} \quad (3.33)$$

where the coefficients κ_i , $i = 1, 2, 3, 4$ are functions of T, μ, χ . Once we reduce this covariant partition function on the time circle and compare it with (3.31) we readily identify

$$\kappa_1 = \frac{T}{T_0} (m_\omega - A_0 m_B), \quad \kappa_2 = -m_B, \quad \kappa_3 = -\beta_1, \quad \kappa_4 = \frac{T_0}{T} \beta_2. \quad (3.34)$$

These relations can be inverted in order to express the surface coefficients (3.32) in terms of the coefficients κ_i leading to the identifications

$$\mathfrak{a}_1 = -\kappa_3, \quad \mathfrak{a}_5 = -\kappa_4, \quad \mathfrak{b}_1 = \kappa_1, \quad \mathfrak{b}_3 = \kappa_2, \quad (3.35)$$

while the remaining transport coefficients must vanish.

3.2.3 Surface currents and thermodynamics

In this section we combine the surface contributions from both the parity-even sector (2.12) (by means of the coefficients α_1 and α_2) and the parity-odd sector (3.21), including the effects of the exterior charged fluid partition function (3.15), which are accounted for by replacing $m_\omega \rightarrow m_\omega - M_\omega$ and $m_B \rightarrow m_B - M_B$ in (3.21). Using (3.31), (3.11) and (2.17), the surface energy-momentum tensor, charge current and entropy current for 2+1 superfluid bubbles takes the form

$$\begin{aligned} T_{(s)}^{\mu\nu} &= (\mathcal{E} - \mathcal{Y}) u^\mu u^\nu - \mathcal{Y} (\mathcal{G}^{\mu\nu} - n^\mu n^\nu) + \mathcal{F} \tilde{\xi}^\mu \tilde{\xi}^\nu + 2\mathcal{U} u^{(\mu} \bar{n}^{\nu)}, \\ J_{(s)}^\mu &= \mathcal{Q} u^\mu - \mathcal{F} \tilde{\xi}^\mu + \mathcal{V} \bar{n}^\mu, \\ J_{(s)\text{ent}}^\mu &= \mathcal{S} u^\mu + \frac{1}{T} (\mathcal{U} - \mu \mathcal{V}) \bar{n}^\mu, \end{aligned} \quad (3.36)$$

where $\bar{n}^\mu = \epsilon^{\mu\rho\lambda} u_\rho n_\lambda$. After imposing the on-shell condition $\partial\mathcal{C}/\partial\lambda = 0$, the various transport coefficients are given by²⁹

$$\begin{aligned} \mathcal{E} &= -\mathcal{C} + T \frac{\partial\mathcal{C}}{\partial T} + \mu \frac{\partial\mathcal{C}}{\partial\mu} + \frac{\mu^2}{\bar{\lambda}} \frac{\partial\mathcal{C}}{\partial\bar{\lambda}} + (\lambda f_1 - \bar{\lambda} \kappa_3), \quad \mathcal{Y} = -\mathcal{C}, \quad \mathcal{F} = -\frac{1}{\bar{\lambda}} \frac{\partial\mathcal{C}}{\partial\bar{\lambda}}, \\ \mathcal{Q} &= \frac{\partial\mathcal{C}}{\partial\mu} + \mu \frac{1}{\bar{\lambda}} \frac{\partial\mathcal{C}}{\partial\bar{\lambda}} + (\lambda \alpha_2 - \bar{\lambda} \kappa_4), \quad \mathcal{S} = \frac{\partial\mathcal{C}}{\partial T} + \frac{\lambda}{T} (\alpha_1 - \mu \alpha_2) - \frac{\bar{\lambda}}{T} (\kappa_3 - \mu \kappa_4), \\ \mathcal{U} &= \frac{T}{T_0} (m_\omega - M_\omega) - \mu (m_B - M_B) = \kappa_1, \quad \mathcal{V} = -(m_B - M_B) = \kappa_2. \end{aligned} \quad (3.37)$$

These relations in turn imply the Gibbs-Duhem and Euler relations of thermodynamics at the surface respectively

$$\begin{aligned} d\mathcal{Y} &= -\left(\mathcal{S} - \frac{\lambda}{T} (f_1 - \mu f_2) + \frac{\bar{\lambda}}{T} (\kappa_3 - \mu \kappa_4) \right) dT \\ &\quad - \left(\mathcal{Q} - \lambda f_2 + \bar{\lambda} \kappa_4 \right) d\mu - \mathcal{F} (\mu d\mu - \bar{\lambda} d\bar{\lambda}), \\ \mathcal{E} - \mathcal{Y} &= T\mathcal{S} + \mu\mathcal{Q}. \end{aligned} \quad (3.38)$$

The respective first law of thermodynamics has been discussed in appendix A.

We see that the surface thermodynamics of 2 + 1 dimensional superfluid bubbles has new features compared to their 3 + 1 dimensional counterparts. In particular, not only does the parity-even coefficients f_1, f_2 directly affect the surface thermodynamics as in the 3 + 1 the dimensional case, but even the parity-odd coefficients κ_3, κ_4 have an effect, in exactly the same way as the coefficients f_1, f_2 do.

²⁹One can easily obtain the contribution of the coefficients κ_3 and κ_4 to the surface entropy density \mathcal{S} by performing a variation of the integrand of (3.33) with respect to T .

4 Galilean stationary superfluid bubbles in 3+1 dimensions

In this section, we analyze the surface currents for stationary bubbles of a 3 + 1 dimensional Galilean superfluid immersed in an ordinary fluid. Any appropriately defined non-relativistic limit of the relativistic currents worked out in section 2, should be a special case of Galilean superfluids. In this sense, Galilean superfluids can provide us with a general understanding of the respective non-relativistic physics. As in the relativistic case, our primary focus here will be the surface currents. A complete analysis of the bulk currents in this case has already been provided in [10].

Our basic setup has been thoroughly described in section 1.2. At first, we have to work out the constitutive relations for a null superfluid in 4 + 1 dimensions using an equilibrium partition function, and then perform a null reduction on it in order to obtain the Galilean results. We shall report the results of this section in a slightly different notation compared to the relativistic case, so as to be closer to those usually used in the non-relativistic superfluid literature. Let us define the superfluid potential $\mu_s = -\frac{1}{2}\xi^M \xi_M = -\mu + \mu_n + \hat{\mu}_s$, where $\hat{\mu}_s = -\frac{1}{2}\zeta^k \zeta_k$, in addition to the usual zero derivative scalars: temperature $T = e^{-\sigma} T_0$, chemical potential $\mu = e^{-\sigma} A_0$ and mass chemical potential $\mu_n = e^{-\sigma} B_0$. We will denote $\nu = \mu/T$ and $\nu_n = \mu_n/T$. It will also be useful to define a boundary superfluid velocity projected on the surface, $\tilde{\xi}^M = (\mathcal{G}^{MN} - n^M n^N) \xi_N$ and an associated potential, $\tilde{\mu}_s = -\frac{1}{2}\tilde{\xi}^M \tilde{\xi}_M = -\mu + \mu_n + \hat{\mu}_s + \frac{1}{2}\lambda^2$ with $\lambda = n_M \xi^M = n_i \zeta^i$, as before.

Up to first order in the bulk and ideal order at the surface, the partition function can be written in terms of the shape-field and background data (1.19) as follows

$$\begin{aligned}
 W = & \int d^3x \sqrt{g} \theta(f) \frac{e^\sigma}{T_0} \left(P_{(b)} - f_1 \zeta^i \partial_i \sigma + e^{-\sigma} f_2 \zeta^i \partial_i A_0 + e^{-\sigma} f_3 \zeta^i \partial_i B_0 \right. \\
 & \left. + T_0 e^{-\sigma} (g_1 + g_2) \epsilon^{ijk} \zeta_i \partial_j B_k + T_0 e^{-\sigma} g_2 \epsilon^{ijk} \zeta_i \partial_j A_k + T_0 (g_1 e^{-\sigma} B_0 + g_2 e^{-\sigma} A_0 - g_3) \epsilon^{ijk} \zeta_i \partial_j a_k \right) \\
 & + \int d^3x \sqrt{g} \tilde{\delta}(f) \frac{e^\sigma}{T_0} \mathcal{C} + \int d^3x \sqrt{g} \theta(-f) \frac{e^\sigma}{T_0} P_{(e)}. \tag{4.1}
 \end{aligned}$$

In order to obtain the thermodynamics in the conventional notation, we consider $P = P(T, \mu, \mu_n, \mu_s)$, while the rest of the bulk transport coefficients f_i and g_i are considered to be functions of $\{T, \nu, \nu_n, \hat{\mu}_s\}$. On the other hand, for the surface tension we consider $\mathcal{C} = \mathcal{C}(T, \mu, \mu_n, \tilde{\mu}_s, \lambda)$. Since outside the bubble there is an ordinary fluid, it cannot depend on the superfluid variables ζ_i or μ_s , leading to no possible terms which can be written at first order. Furthermore, $P_{(e)} = P_{(e)}(T, \mu, \mu_n)$ is independent of μ_s .

We start with the ϕ equations of motion obtained by varying the partition function (4.1) with respect to ϕ

$$\begin{aligned}
 \theta(f) \left[D_i \left(\frac{1}{T} \frac{\partial P_{(b)}}{\partial \mu_s} \zeta^i + \frac{1}{T^2} \frac{\partial f_1}{\partial \hat{\mu}_s} \zeta^i \zeta^j \partial_j T - \frac{1}{T^2} f_1 g^{ij} \partial_j T + \frac{\partial f_2}{\partial \hat{\mu}_s} \zeta^i \zeta^j \partial_j \nu - f_2 g^{ij} \partial_j \nu \right. \right. \\
 \left. \left. + \frac{\partial f_3}{\partial \hat{\mu}_s} \zeta^i \zeta^j \partial_j \nu_n - f_3 g^{ij} \partial_i \nu_n + \frac{\partial (g_1 + g_2)}{\partial \hat{\mu}_s} \zeta^i \epsilon^{ajk} \zeta_a \partial_j B_k - (g_1 + g_2) \epsilon^{ijk} \partial_j B_k \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\partial g_2}{\partial \hat{\mu}_s} \zeta^i \epsilon^{ajk} \zeta_a \partial_j A_k - g_2 \epsilon^{ijk} \partial_j A_k + e^\sigma \left(\frac{\partial g_1}{\partial \hat{\mu}_s} \mu_n + \frac{\partial g_2}{\partial \hat{\mu}_s} \mu - \frac{\partial g_3}{\partial \hat{\mu}_s} \right) \zeta^i \epsilon^{ajk} \zeta_a \partial_j a_k \\
 & - e^\sigma (g_1 \mu_n + g_2 \mu - g_3) \epsilon^{ijk} \partial_j a_k \Big] , \tag{4.2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\delta}(f) \Big[& T \tilde{D}_i \left(\frac{1}{T} \frac{\partial \mathcal{C}}{\partial \lambda} n^i - \frac{1}{T} \frac{\partial \mathcal{C}}{\partial \hat{\mu}_s} \tilde{\zeta}^i \right) + \lambda \frac{\partial P_{(b)}}{\partial \mu_s} + \frac{1}{T} \frac{\partial f_1}{\partial \hat{\mu}_s} \lambda \zeta^i \partial_i T - \frac{1}{T} f_1 n^i \partial_i T + T \frac{\partial f_2}{\partial \hat{\mu}_s} \lambda \zeta^i \partial_i \nu \\
 & - T f_2 n^i \partial_i \nu + T \frac{\partial f_3}{\partial \hat{\mu}_s} \lambda \zeta^i \partial_i \nu_n - T f_3 n^i \partial_i \nu_n + T \frac{\partial (g_1 + g_2)}{\partial \hat{\mu}_s} \lambda \epsilon^{ijk} \zeta_i \partial_j B_k \\
 & - T (g_1 + g_2) \epsilon^{ijk} n_i \partial_j B_k + T \frac{\partial g_2}{\partial \hat{\mu}_s} \lambda \epsilon^{ijk} \zeta_i \partial_j A_k - T g_2 \epsilon^{ijk} n_i \partial_j A_k \\
 & + T_0 \left(\frac{\partial g_1}{\partial \hat{\mu}_s} \mu_n + \frac{\partial g_2}{\partial \hat{\mu}_s} \mu - \frac{\partial g_3}{\partial \hat{\mu}_s} \right) \lambda \epsilon^{ijk} \zeta_i \partial_j a_k - T_0 (g_1 \mu_n + g_2 \mu - g_3) \epsilon^{ijk} n_i \partial_j a_k \Big] , \\
 \tilde{\delta}'(f) \left[\frac{\partial \mathcal{C}}{\partial \lambda} \right] & = 0 . \tag{4.3}
 \end{aligned}$$

Taking a variation of the partition function (4.1) and using the variational formulae (C.4), we can read off the surface currents (for a discussion on the bulk currents see [10]), after using the $\tilde{\delta}'(f)$ equation of motion of ϕ

$$\begin{aligned}
 T_{(s)_{--}} &= \mathcal{R}_n + \mathcal{R}_s, & T_{(s)}^i{}_{-} &= -g_1 \bar{n}^i - \mathcal{R}_s \tilde{\zeta}^i, & T_{(s)}^{ij} &= -h^{ij} \mathcal{Y} + R_{s,\partial} \tilde{\zeta}^i \tilde{\zeta}^j, \\
 T_{(s)_{0-}} &= e^\sigma (\mathcal{E} - \mu_n \mathcal{R}_n) - \xi_0 \mathcal{R}_s, & T_{(s)}^i{}_0 &= -e^\sigma (g_3 - g_1 \mu_n) \bar{n}^i + \xi_0 \mathcal{R}_s \tilde{\zeta}^i, \\
 J_{(s)-} &= -\mathcal{Q} + \mathcal{R}_s, & J_{(s)}^i &= -\mathcal{R}_s \tilde{\zeta}^i + g_2 \bar{n}^i, \tag{4.4}
 \end{aligned}$$

where $h^{ij} = g^{ij} - n^i n^j$, $\tilde{\zeta}^i = h^{ij} \zeta_j$, $\bar{n}^i = T \epsilon^{ijk} \zeta_j n_k$, and we have defined the surface first law of thermodynamics and the Euler relation

$$\begin{aligned}
 d(\mathcal{E} - \lambda f_1) &= T d \left(\mathcal{S} - \frac{\lambda}{T} (f_1 - \mu f_2 - \mu_n f_3) \right) + \mu_n d(\mathcal{R}_n - \lambda f_3) + \mu d(\mathcal{Q} - \lambda f_2) - \mathcal{R}_s d\tilde{\mu}_s, \\
 \mathcal{E} - \mathcal{Y} &= T \mathcal{S} + \mu \mathcal{Q} + \mu_n \mathcal{R}_n, \tag{4.5}
 \end{aligned}$$

where $\mathcal{Y} = -\mathcal{C}$ is the surface tension. Finally, the equation of motion of f yields the Young-Laplace equation for Galilean/null superfluids

$$\begin{aligned}
 P_{(b)} - P_{(e)} + \frac{1}{T} f_1 \zeta^i \partial_i T + T f_2 \zeta^i \partial_i \nu + T f_3 \zeta^i \partial_i \nu_n + T (g_1 + g_2) \epsilon^{ijk} \zeta_i \partial_j B_k + T g_2 \epsilon^{ijk} \zeta_i \partial_j A_k \\
 + T_0 (\mu_n g_1 + \mu g_2 - g_3) \epsilon^{ijk} \zeta_i \partial_j a_k + T D_i \left(\frac{1}{T} \mathcal{C} n^i + \frac{\lambda}{T} \mathcal{R}_s \tilde{\zeta}^i \right) = 0. \tag{4.6}
 \end{aligned}$$

The same equation can also be obtained by projecting the surface energy-momentum conservation equation along n_M (see appendix B). After properly covariantizing the expressions (4.4), and using a hydrodynamic frame suitable for the equilibrium partition function

$$u^M = e^{-\sigma} (B_0, 1, 0, 0), \quad T = e^{-\sigma} T_0, \quad \mu = e^{-\sigma} A_0, \quad \mu_n = e^{-\sigma} B_0, \tag{4.7}$$

we have the surface currents

$$\begin{aligned}
 T_{(s)}^{MN} &= \mathcal{R}_n u^M u^N + 2(\mathcal{E} - \mathcal{C}) u^{(M} V^{N)} - \mathcal{C} (\mathcal{G}^{MN} - n^M n^N) + \mathcal{R}_s \tilde{\zeta}^M \tilde{\zeta}^N + 2g_1 u^{(M} \bar{n}^{N)} + 2g_3 V^{(M} \bar{n}^{N)}, \\
 J_{(s)}^M &= \mathcal{Q} u^M - \mathcal{R}_s \tilde{\zeta}^M + g_2 \bar{n}^M, \tag{4.8}
 \end{aligned}$$

where $\bar{n}^M = T\epsilon^{MNRST}V_N u_R \xi_S n_T$. The respective thermodynamics is given by (4.5). Note that the most generic form of the constitutive relations at ideal order (transverse to n^M) could have contained three more terms proportional to $u^{(M}\tilde{\xi}^{N)}$, $V^{(M}\tilde{\xi}^{N)}$ and $\tilde{\xi}^{(M}\bar{n}^{N)}$ in the energy-momentum tensor, making a total of 12 independent terms. The equilibrium partition function fixes these 12 coefficients in terms of a boundary function \mathcal{C} and 6 first order bulk coefficients f_i, g_i .

Finally, upon performing the null reduction, the leading order surface currents and densities for a 3 + 1 dimensional Galilean superfluid can be obtained as

$$\text{Mass Density: } \rho_{(s)} = \mathcal{R}_n + \mathcal{R}_s, \quad (4.9a)$$

$$\text{Mass Current: } \rho_{(s)}^i = \mathcal{R}_n u^i + \mathcal{R}_s \tilde{\xi}^i + g_1 \bar{n}^i, \quad (4.9b)$$

$$\text{Stress Tensor: } t_{(s)}^{ij} = \mathcal{R}_n u^i u^j - \mathcal{Y} h^{ij} + \mathcal{R}_s \tilde{\xi}^i \tilde{\xi}^j + 2g_1 u^{(i} \bar{n}^{j)}, \quad (4.9c)$$

$$\text{Energy Density: } \epsilon_{(s)} = \mathcal{E} + \mathcal{R}_s \tilde{\mu}_s + \frac{1}{2} \mathcal{R}_n u^k u_k + \frac{1}{2} \mathcal{R}_s \tilde{\xi}^k \tilde{\xi}_k + g_1 \bar{n}^i u_i,$$

$$\begin{aligned} \text{Energy Current: } \epsilon_{(s)}^i &= u^i \left(\mathcal{E} - \mathcal{Y} + \frac{1}{2} \mathcal{R}_n u^k u_k + g_1 \bar{n}^i u_j \right) + \mathcal{R}_s \tilde{\xi}^i \left(\frac{1}{2} \tilde{\xi}^k \tilde{\xi}_k + \tilde{\mu}_s \right) \\ &\quad + \left(g_3 + \frac{1}{2} g_1 u^k u_k \right) \bar{n}^i, \end{aligned} \quad (4.9d)$$

$$\text{Charge Density: } q_{(s)} = \mathcal{Q} - \mathcal{R}_s, \quad (4.9e)$$

$$\text{Charge Current: } q_{(s)}^i = \mathcal{Q} u^i - \mathcal{R}_s \tilde{\xi}^i + g_2 \bar{n}^i. \quad (4.9f)$$

It is interesting to contrast these results with those in the bulk, as reported by [10]. Not only there are new terms in the leading order Galilean constitutive relations, but some of them are parity-odd as well. Furthermore, all these new terms are completely determined in terms of the first order bulk transport coefficients. In fact, since all the first order stationary bulk coefficients appear in the surface constitutive relations, they can, in principle, be measured by performing carefully designed experiments on the surface of the superfluid.

5 Surface dynamics

In this section, we study the consequences of a non-trivial time dependence of the shape-field on the surface. Once we relax the assumption of stationarity, we cannot deduce the constitutive relations of a (super)fluid through an equilibrium partition function, as we did in section 2 and section 3. Therefore, we have to resort to the second law of thermodynamics to constrain and understand the full time-dependent dynamics. Hence, we first analyze the surface entropy current at ideal order in section 5.1, to understand the structure of the equations governing the surface dynamics. With this understanding, in section 5.2 we study linearized fluctuations on the surface and its relation with the fluctuations in the bulk, both for an ordinary fluid and a superfluid.

5.1 Surface entropy current analysis at zero derivative order

5.1.1 Surface entropy current for ordinary fluids

Before proceeding to the superfluid case, we study the entropy current and the consequences of the second law of thermodynamics for ordinary fluids in the presence of a surface. Once we give up the assumption of stationarity, the first aspect of surface dynamics we would like to understand is what determines the normal component of the fluid velocity $u^\mu n_\mu$ at the surface. In the stationary case, this normal component vanishes as $\tilde{K}^\mu = e^\sigma u^\mu$ is a Killing vector field.³⁰ The second aspect of surface dynamics we would like to understand is what determines the equation of motion for the shape-field f , since it is not clear a priori if the normal component of the surface energy-momentum conservation continues to serve as a proxy for the equation of motion of f in non-equilibrium situations. In this section, we will try to answer both these questions and demonstrate that they are interrelated.

As mentioned above, in the analysis of equilibrium partition functions, $u^\mu n_\mu$ was zero by construction. In fact, this condition served as one of the boundary conditions for solving the bulk fluid equations (see section 1 and [8] for more details). However, as we move away from stationarity, the status of $u^\mu n_\mu$ is not clear a priori and we need a principle to determine it. In order to address this problem, it is extremely useful to remember the analogy between the shape-field f and the superfluid phase ϕ , both being a consequence of a spontaneously broken symmetry. Momentarily, if we take this analogy seriously then $u^\mu n_\mu$ would correspond to $u^\mu \xi_\mu$ in the case of superfluids. Now, as we know, $u^\mu \xi_\mu$ is not an independent variable in superfluid dynamics. In fact, it is given by the chemical potential μ [6] at leading order and receives further corrections at higher orders, as determined by the second law of thermodynamics [4]. As noted in [27], the generalized Josephson equation $u^\mu \xi_\mu = \mu + \mu_{\text{diss}}$ can be derived using an entropy current analysis. It was also observed in [27] that in equilibrium, and in a hydrodynamic frame chosen appropriately for equilibrium, the equation $u^\mu \xi_\mu = \mu + \mu_{\text{diss}}$ reduces to $\mu_{\text{diss}} = 0$, which can be identified as the equation of motion for ϕ following from the respective equilibrium partition function [12]. Therefore, the Josephson equation can be thought of as the equation of motion for ϕ outside equilibrium. This gives us an important clue for the case of the shape-field: $u^\mu n_\mu$ should also be determined by the second law of thermodynamics in terms of other fluid variables, and the respective determining relation should be the equation of motion for f outside equilibrium. For this purpose, let us define

$$u^\mu n_\mu = \gamma + \gamma_{\text{diss}}, \quad (5.1)$$

where γ is the zeroth order value of $u^\mu n_\mu$ and γ_{diss} contains the higher derivative corrections. It is definitely possible to choose a hydrodynamic frame where $\gamma_{\text{diss}} = 0$, just as it is possible

³⁰This simply follows as $u^\mu n_\mu \propto \tilde{K}^\mu \partial_\mu f = \mathcal{L}_{\tilde{K}} f = 0$. Another way to argue this is that on the surface we have $d+3$ undetermined variables in $d+1$ dimensions: $T|_{f=0}$, $\mu|_{f=0}$, d components of $u^\mu|_{f=0}$ (including $u^\mu n_\mu$) and f . Since we only have $d+2$ conservation laws, for the system to be solvable, there must be another relation among these variables. Later in this section, we will show that the second law of thermodynamics forces such a relation to imply $u^\mu n_\mu = 0$ in equilibrium. This goes on to show that $u^\mu n_\mu$ should not be treated as an independent thermodynamic variable at the surface, as was done in [9].

to choose a frame where $\mu_{\text{diss}} = 0$ in the case of superfluids. However, such a frame would not correspond to the more standard frame choices like the Landau frame, neither would it be a generalization of the equilibrium frame defined in section 1.

Let us now proceed to analyze the structure of the divergence of the surface entropy current. The bulk energy-momentum tensor and entropy current have the well known form

$$\begin{aligned} T_{(b)}^{\mu\nu} &= (E + P) u^\mu u^\nu + P \mathcal{G}^{\mu\nu} + \Pi_{(b)}^{\mu\nu}, \\ J_{(b)\text{ent}}^\mu &= S u^\mu + \Upsilon_{(b)\text{ent}}^\mu, \quad \Upsilon_{(b)\text{ent}}^\mu = -\frac{u_\nu}{T} \Pi_{(b)}^{\mu\nu} + \Upsilon_{(b)\text{new}}^\mu, \end{aligned} \quad (5.2)$$

where $\Pi_{(b)}^{\mu\nu}$ and $\Upsilon_{(b)\text{new}}^\mu$ are higher derivative corrections, which can be found, for example, in [31]. It is interesting to note that $\Upsilon_{(b)\text{new}}^\mu$ does not receive any first order corrections [31]. On the other hand, the ideal order surface currents are given by³¹

$$\begin{aligned} T_{(s)}^{\mu\nu} &= (\mathcal{E} - \mathcal{Y}) u^\mu u^\nu - \mathcal{Y} (\mathcal{G}^{\mu\nu} - n^\mu n^\nu) + \Pi_{(s)}^{\mu\nu}, \\ J_{(s)\text{ent}}^\mu &= \mathcal{S} u^\mu + \Upsilon_{(s)\text{ent}}^\mu, \quad \Upsilon_{(s)\text{ent}}^\mu = -\frac{u_\nu}{T} \Pi_{(s)}^{\mu\nu} + \Upsilon_{(s)\text{new}}^\mu, \end{aligned} \quad (5.3)$$

where the $\mathcal{Y}, \mathcal{E}, \mathcal{S}$ are the surface tension, energy density and entropy density on the surface respectively, and $\Pi_{(s)}^{\mu\nu}, \Upsilon_{(s)\text{new}}^\mu$ are higher derivative corrections. These derivative corrections will not play any significant role in our discussion below, but we retain them for completeness. The surface conservation equation projected along the fluid velocity takes the form

$$\begin{aligned} u_\nu \tilde{\nabla}_\mu T_{(s)}^{\mu\nu} - u_\nu n_\mu T_{(b)}^{\mu\nu} &= -u^\mu \partial_\mu \mathcal{E} - (\mathcal{E} - \mathcal{Y}) \tilde{\nabla}_\mu u^\mu \\ &+ u^\mu n_\mu \left(\nabla_\mu (\mathcal{Y} n^\mu) + E \right) + u_\nu \left(\tilde{\nabla}_\mu \Pi_{(s)}^{\mu\nu} - n_\mu \Pi_{(b)}^{\mu\nu} \right) = 0. \end{aligned} \quad (5.4)$$

Now, the divergence of the entropy current on the boundary, including the possible entropy exchange with the bulk, must be positive semi-definite. This condition upon using the equation of motion (5.4) simplifies to

$$\begin{aligned} \tilde{\nabla}_\mu J_{(s)\text{ent}}^\mu - n_\mu J_{(b)\text{ent}}^\mu &= \frac{u^\mu n_\mu}{T} \left(\nabla_\mu (\mathcal{Y} n^\mu) - P \right) - \Pi_{(s)}^{\mu\nu} \nabla_\mu \left(\frac{u_\nu}{T} \right) \\ &+ \tilde{\nabla}_\mu \Upsilon_{(s)\text{new}}^\mu - n_\mu \Upsilon_{(b)\text{new}}^\mu \geq 0, \end{aligned} \quad (5.5)$$

where we have made use of the Euler relation $E + P = TS$ and $\mathcal{E} - \mathcal{Y} = TS$, as well as of the first law $dE = TdS$ and $d\mathcal{E} = Td\mathcal{S}$. Up to first order in the bulk and ideal order at the surface, (5.5) implies that

$$\frac{u^\nu n_\nu}{T} \left(\nabla_\mu (\mathcal{Y} n^\mu) - P \right) \geq 0. \quad (5.6)$$

³¹Note there that we have not assumed the tangentiality conditions $T_{(s)}^{\mu\nu} n_\mu = J_{(s)\text{ent}}^\mu n_\mu = 0$ on the surface energy-momentum tensor and entropy current, since we wish to derive such tangentiality conditions at leading order from the entropy current analysis. Furthermore, note that in (5.3) we have not considered terms of the form $\theta_1 u^{(\mu} n^{\nu)}$ and $\theta_2 n^\mu n^\mu$ in the surface energy-momentum tensor neither have we consider a term proportional to $\theta_3 n^\mu$ in the surface entropy current. In full generality, such terms must be taken into account but for clarity of presentation we have not introduced them. In any case, the second law of thermodynamics ultimately implies that $\theta_1 = \theta_2 = \theta_3 = 0$.

The condition (5.6) must hold for an arbitrary fluid configuration, including the ones for which the term inside the bracket may have a negative sign. This implies that at leading order $u^\mu n_\mu$ must vanish, that is

$$\gamma = 0. \quad (5.7)$$

This is the first important conclusion of this section.

As we move to higher orders, other terms in (5.5) become important for this analysis. An important noteworthy structural feature in (5.5) is the fact that the only term which contains the bulk transport coefficients is the last term $n_\mu \Upsilon_{(b)\text{new}}^\mu$. This immediately implies that only the transport coefficients that arise in $\Upsilon_{(b)\text{new}}^\mu$ are the ones that may be related to the surface transport coefficients. An interesting observation can be made, if we focus on perfect fluid bubbles, i.e. $\Pi_{(b)}^{\mu\nu} = \Upsilon_{(b)\text{new}}^\mu = \Pi_{(s)}^{\mu\nu} = \Upsilon_{(s)\text{new}}^\mu = 0$. For this choice, (5.5) simply implies (after setting $\gamma = 0$)

$$\frac{\gamma_{\text{diss}}}{T} \left(\nabla_\mu (\mathcal{Y} n^\mu) - P \right) \geq 0, \quad (5.8)$$

which has a solution

$$\gamma_{\text{diss}} = \varsigma \left(\nabla_\mu (\mathcal{Y} n^\mu) - P \right), \quad \text{with,} \quad \varsigma \geq 0. \quad (5.9)$$

Here ς has the status of a dissipative transport coefficient. The respective f equation of motion away from equilibrium is then

$$u^\mu n_\mu = \varsigma \left(\nabla_\mu (\mathcal{Y} n^\mu) - P \right). \quad (5.10)$$

In equilibrium $u^\mu n_\mu = 0$, and consequently $\gamma_{\text{diss}} = 0$ implies the perfect ordinary fluid Young-Laplace equation, $\nabla_\mu (\mathcal{Y} n^\mu) = P$. In order to see this exactly, note that the Young-Laplace equation, defined as the normal component of the surface energy-momentum conservation equation is $-T_{(s)}^{\mu\nu} K_{\mu\nu} = T_b^{\mu\nu} n_\mu n_\nu$ [8], which at ideal order implies that

$$\mathcal{Y} K - T \frac{\partial \mathcal{Y}}{\partial T} n_\mu \mathfrak{a}^\mu = P + \mathcal{O}(\gamma_{\text{diss}}^2, \partial \gamma_{\text{diss}}). \quad (5.11)$$

Here $K_{\mu\nu} = \nabla_{(\mu} n_{\nu)}$ is the extrinsic curvature tensor of the surface, $\nabla_\mu n^\mu = \mathcal{G}^{\mu\nu} K_{\mu\nu} = K$ is the mean extrinsic curvature and $\mathfrak{a}^\mu = u^\nu \nabla_\nu u^\mu$ is the fluid acceleration. Using the fact that in equilibrium $n^\mu \partial_\mu T = -T \mathfrak{a}^\mu n_\mu$, the equivalence between (5.8) and (5.11) in equilibrium immediately follows.

However, under the assumption of perfect fluid bubbles, for which (5.8) applies, one may use the fact that, on-shell, the normal component of the vector bulk equation of motion implies that $n^\mu \partial_\mu T = -T \mathfrak{a}^\mu n_\mu$ at the surface. Therefore, ignoring higher order corrections, the Young-Laplace equation implies that on-shell $u^\mu n_\mu = \gamma_{\text{diss}} = 0$ for perfect fluids, even away from equilibrium.³² When we include first order terms in the bulk, i.e. $\Pi_{(b)}^{\mu\nu} = -\eta \sigma^{\mu\nu} - \zeta \Theta (\mathcal{G}^{\mu\nu} + u^\mu u^\nu)$, where $\sigma_{\mu\nu}$ and Θ are the fluid shear tensor and expansion respectively, Young-Laplace equation modifies as

$$\mathcal{Y} K - T \frac{\partial \mathcal{Y}}{\partial T} n_\mu \mathfrak{a}^\mu = P - \eta \sigma^{\mu\nu} n_\mu n_\nu - \zeta \Theta + \mathcal{O}(\gamma_{\text{diss}}^2, \partial \gamma_{\text{diss}}). \quad (5.12)$$

³²This equivalence holds on-shell but not off-shell in the sense of [32].

On the other hand, the f equation of motion in (5.8) remains unchanged, since $J_{(b)\text{new}}^\mu$ is known to be zero at first order for ordinary fluids. Hence for onshell configurations, we can rewrite (5.12) as

$$u^\mu n_\mu = \gamma_{\text{diss}} = -\varsigma (\eta \sigma^{\mu\nu} n_\mu n_\nu + \zeta \Theta) + \mathcal{O}(\gamma_{\text{diss}}^2, \partial \gamma_{\text{diss}}). \quad (5.13)$$

We can see that upon including derivative corrections, $u^\mu n_\mu = \gamma_{\text{diss}} \neq 0$ away from equilibrium. We would like to note that, upon including further higher order corrections, either in the bulk or at the surface, and hence moving further away from the simplified case of perfect fluid bubbles, we might expect (5.8) as well as (5.11), to be modified.

5.1.2 Surface entropy current for 3+1 dimensional superfluids

Having understood the behaviour of $u^\mu n_\mu$ for neutral fluids, in this subsection we will explore the similar entropy current analysis for superfluids with a surface. We will demonstrate that the first law of thermodynamics in 3 + 1 dimensions modifies like (2.20), and includes contributions from the first order bulk transport coefficients α_1 and α_2 . We shall also exhibit, that the second law of thermodynamics puts no constraints on $n_\mu \xi^\mu$ at the interface, in contrast to the normal component of the fluid velocity $u^\mu n_\mu$ which is set to zero at ideal order.

For superfluids, the bulk currents take the well known form

$$\begin{aligned} T_{(b)}^{\mu\nu} &= (E + P) u^\mu u^\nu + P \mathcal{G}^{\mu\nu} + F \xi^\mu \xi^\nu + \Pi_{(b)}^{\mu\nu}, \\ J_{(b)}^\mu &= Q u^\mu - F \xi^\mu + \Upsilon_{(b)}^\mu, \\ J_{(b)\text{ent}}^\mu &= S u^\mu + \Upsilon_{(b)\text{ent}}^\mu, \quad \Upsilon_{(b)\text{ent}}^\mu = -\frac{u_\nu}{T} \Pi_{(b)}^{\mu\nu} - \frac{\mu}{T} J_{(b)}^\mu + \Upsilon_{(b)\text{new}}^\mu. \end{aligned} \quad (5.14)$$

Here, the leading order coefficients follow the usual superfluid thermodynamics $E + P = ST + \mu Q$, $dP = SdT + Qd\mu + \frac{1}{2}Fd\chi$. In our analysis here, the first order corrections to the bulk entropy current $\Upsilon_{(b)\text{new}}^\mu$ will play an important role. The first order terms in $\Upsilon_{(b)\text{new}}^\mu$ were obtained in [4] and the coefficients were related to those in the partition function (2.11) in [12]. Setting $\alpha_3 = 0$ as in section 2.1, $\Upsilon_{(b)\text{new}}^\mu$ reads (see [10, 12])³³

$$\begin{aligned} \Upsilon_{(b)\text{new}}^\mu &= \nabla_\nu \left(c_1 u^{[\mu} \xi^{\nu]} + c_2 \epsilon^{\mu\nu\rho\sigma} u_\rho \xi_\sigma \right) \\ &+ \left(\frac{f_1}{T} 2u^{[\mu} \xi^{\nu]} \frac{1}{T} \partial_\nu T + \frac{u^\mu \xi_\mu - \mu}{T} \mathcal{O}_\chi(f_1)^{\mu\nu} \frac{1}{T} \partial_\nu T \right) \end{aligned} \quad (5.16)$$

³³At first order, the only contribution to $\Upsilon_{(b)\text{new}}^\mu$ comes from the equilibrium sector and is obtained as follows [27]: write down the most general scalar \mathcal{L} made out of first order data that survives in equilibrium (it can be thought of as a covariant version of the partition function), and perform a variation keeping the fluid variables constant

$$\frac{1}{\sqrt{\mathcal{G}}} \delta(\sqrt{\mathcal{G}} \mathcal{L}) = \frac{1}{2} T_{\mathcal{L}}^{\mu\nu} \delta \mathcal{G}_{\mu\nu} + J_{\mathcal{L}}^\mu \delta \mathcal{A}_\mu + K_{\mathcal{L}} \delta \varphi + Y_{\mathcal{L}} \delta \phi + \nabla_\mu \Theta_{\mathcal{L}}^\mu, \quad (5.15)$$

where $\nabla_\mu \Theta_{\mathcal{L}}^\mu$ is a total derivative gained via differentiation by parts. Having done that, in an appropriate hydrodynamic frame which is a generalization of the equilibrium frame, we have $\Upsilon_{(b)\text{new}}^\mu = \frac{1}{T} \mathcal{L} u^\mu - \Theta_{\mathcal{L}}^\mu$. In fact, $T_{\mathcal{L}}^{\mu\nu}$, $J_{\mathcal{L}}^\mu$, $K_{\mathcal{L}}$ and $Y_{\mathcal{L}}$ are the first order equilibrium energy-momentum tensor, charge current, ϕ variation and f variation respectively in the bulk, gained via the equilibrium partition function.

$$\begin{aligned}
 & + \left(\frac{f_2}{T} 2u^{[\mu} \xi^{\nu]} T \partial_\nu \nu + \frac{u^\mu \xi_\mu - \mu}{T} \mathcal{O}_\chi (f_2)^{\mu\nu} T \partial_\nu \nu \right) \\
 & + \left(\frac{g_1}{T} \epsilon^{\mu\nu\rho\sigma} \zeta_\nu \left(\partial_\rho u_\sigma + u_\sigma \frac{1}{T} \partial_\rho T \right) + \frac{u^\mu \xi_\mu - \mu}{T} \mathcal{O}_\chi (g_1)^\mu_\alpha \epsilon^{\alpha\nu\rho\sigma} u_\nu \partial_\rho u_\sigma \right) \\
 & + \left(\frac{g_2}{T} \epsilon^{\mu\nu\rho\sigma} \zeta_\nu \left(\frac{1}{2} \mathcal{F}_{\rho\sigma} + u_\sigma T \partial_\rho \nu \right) + \frac{1}{2} \frac{u^\mu \xi_\mu - \mu}{T} \mathcal{O}_\chi (g_2)^\mu_\alpha \epsilon^{\alpha\nu\rho\sigma} u_\nu \mathcal{F}_{\rho\sigma} \right),
 \end{aligned}$$

where $\zeta_\mu = \xi_\mu + (u_\nu \xi^\nu) u_\mu$ and we have defined an operator $\mathcal{O}_\chi(\cdot)^{\mu\nu} = \left(2\zeta^\mu \zeta^\nu \frac{\partial}{\partial \chi} - P^{\mu\nu} \right)$ for clarity. Note that the first term here is a total derivative, i.e. its divergence trivially vanishes, and hence is not important in the bulk. However it might have some non-trivial boundary effects. Following (2.6), the surface currents have the general form³⁴

$$\begin{aligned}
 T_{(s)}^{\mu\nu} &= (\mathcal{E} - \mathcal{Y}) u^\mu u^\nu - \mathcal{Y} (\mathcal{G}^{\mu\nu} - n^\mu n^\nu) + \mathcal{F} \tilde{\xi}^\mu \tilde{\xi}^\nu + 2\mathcal{U} u^{(\mu} \bar{n}^{\nu)} + \Pi_{(s)}^{\mu\nu}, \\
 J_{(s)}^\mu &= \mathcal{Q} u^\mu - \mathcal{F} \tilde{\xi}^\mu + \mathcal{V} \bar{n}^\mu + \Upsilon_{(s)}^\mu, \\
 J_{(s)\text{ent}}^\mu &= \mathcal{S} u^\mu + \Upsilon_{(s)\text{ent}}^\mu, \quad \Upsilon_{(s)\text{ent}}^\mu = -\frac{u_\nu}{T} \Pi_{(s)}^{\mu\nu} - \frac{\mu}{T} J_{(s)}^\mu + \Upsilon_{(s)\text{new}}^\mu.
 \end{aligned} \tag{5.17}$$

where $\bar{n}^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \xi_\rho n_\sigma$. Again, we work out the following two scalar components of the conservation equations

$$\begin{aligned}
 u_\nu \tilde{\nabla}_\mu T_{(s)}^{\mu\nu} - u_\nu n_\mu T_{(b)}^{\mu\nu} - u_\nu \mathcal{F}^{\nu\mu} J_\mu^{(s)} &= -u^\mu \partial_\mu \mathcal{E} - (\mathcal{E} - \mathcal{Y}) \tilde{\nabla}_\mu u^\mu - \frac{1}{2} \mathcal{F} u^\nu \partial_\nu \tilde{\chi} \\
 &\quad - T \tilde{\nabla}_\mu \left(\frac{1}{T} \mathcal{U} \bar{n}^\mu \right) - 2T \mathcal{U} u^{(\mu} \bar{n}^{\nu)} \nabla_\mu \left(\frac{u_\nu}{T} \right) - \mathcal{V} u_\nu \mathcal{F}^{\nu\mu} \bar{n}_\mu \\
 &\quad + u^\nu \xi_\nu \left(\tilde{\nabla}_\mu (\mathcal{F} \tilde{\xi}^\mu) - (n^\mu \xi_\mu) F \right) + u^\mu n_\mu \left(\nabla_\mu (\mathcal{Y} n^\mu - (n^\nu \xi_\nu) \mathcal{F} \tilde{\xi}^\mu) + E \right) \\
 &\quad + u_\nu \left(\tilde{\nabla}_\mu \Pi_{(s)}^{\mu\nu} - n_\mu \Pi_{(b)}^{\mu\nu} \right) - u_\nu \mathcal{F}^{\nu\mu} \Upsilon_\mu^{(s)} = 0, \\
 \tilde{\nabla}_\mu J_{(s)}^\mu - n_\mu J_{(b)}^\mu &= u^\mu \partial_\mu \mathcal{Q} + \mathcal{Q} \tilde{\nabla}_\mu u^\mu - \tilde{\nabla}_\mu (\mathcal{F} \tilde{\xi}^\mu) + n^\mu \xi_\mu F - u^\mu n_\mu \mathcal{Q} + \tilde{\nabla}_\mu (\mathcal{V} \bar{n}^\mu) \\
 &\quad + \left(\tilde{\nabla}_\mu \Upsilon_{(s)}^\mu - n_\mu \Upsilon_{(b)}^\mu \right) = 0.
 \end{aligned} \tag{5.18}$$

Now, it is possible to show that the divergence of the entropy current conservation at the surface reduces to

$$\begin{aligned}
 \tilde{\nabla}_\mu J_{(s)\text{ent}}^\mu - n_\mu J_{(b)\text{ent}}^\mu &= -\frac{u^\mu}{T} \left(\partial_\mu \mathcal{E} - T \partial_\mu \mathcal{S} - \mu \partial_\mu \mathcal{Q} + \frac{1}{2} \mathcal{F} \partial_\mu \tilde{\chi} \right) - \frac{1}{T} (\mathcal{E} - \mathcal{Y} + T \mathcal{S} + \mu \mathcal{Q}) \tilde{\nabla}_\mu u^\mu \\
 &\quad + \frac{u^\nu \xi_\nu - \mu}{T} \left(\tilde{\nabla}_\mu (\mathcal{F} \tilde{\xi}^\mu) - (n^\mu \xi_\mu) F \right) + \frac{u^\mu n_\mu}{T} \left(\nabla_\mu (\mathcal{Y} n^\mu - (n^\nu \xi_\nu) \mathcal{F} \tilde{\xi}^\mu) + E - T \mathcal{S} - \mu \mathcal{Q} \right) \\
 &\quad - \Pi_{(s)}^{\mu\nu} \nabla_\mu \left(\frac{u_\nu}{T} \right) - \frac{1}{T} \Upsilon_{(s)}^\mu (T \nabla_\mu \nu + u^\nu \mathcal{F}_{\nu\mu}) - 2\mathcal{U} u^{(\mu} \bar{n}^{\nu)} \nabla_\mu \left(\frac{u_\nu}{T} \right) \\
 &\quad - \frac{1}{T} \mathcal{V} \bar{n}^\mu (T \nabla_\mu \nu + u^\nu \mathcal{F}_{\nu\mu}) + \tilde{\nabla}_\mu \left(\Upsilon_{(s)\text{new}}^\mu - \frac{1}{T} (\mathcal{U} - \mu \mathcal{V}) \bar{n}^\mu \right) - n_\mu \Upsilon_{(b)\text{new}}^\mu \geq 0.
 \end{aligned} \tag{5.19}$$

Note that we have not imposed the thermodynamics yet, as there are first derivative terms in $J_{(b)\text{new}}^\mu$ which might modify it. Restricting ourselves to first order in the bulk and ideal

³⁴As in the case of ordinary fluids, one must consider terms proportional to n^μ in the surface energy-momentum tensor and currents. However, such terms will be ultimately set to zero by the entropy current analysis and hence we did not consider them here for clarity of presentation.

order at the boundary, this equation modifies to

$$\begin{aligned}
 & -\frac{u^\mu}{T} \left(\partial_\mu (\mathcal{E} - \lambda f_1) - T \partial_\mu \left(\mathcal{S} - \frac{\lambda}{T} (f_1 - \mu f_2) \right) - \mu \partial_\mu (\mathcal{Q} - \lambda f_2) + \frac{1}{2} \mathcal{F} \partial_\mu \tilde{\chi} \right) \\
 & - \frac{1}{T} (\mathcal{E} - \mathcal{Y} + T \mathcal{S} + \mu \mathcal{Q}) \tilde{\nabla}_\mu u^\mu + \frac{u^\nu \xi_\nu - \mu}{T} \mathcal{E}_\phi + \frac{u^\mu n_\mu}{T} \mathcal{E}_f \\
 & - 2(\mathcal{U} - g_1) u^{(\mu} \bar{n}^{\nu)} \nabla_\mu \left(\frac{u_\nu}{T} \right) - \frac{\mathcal{V} - g_2}{T} \bar{n}^\mu (T \partial_\mu \nu + u^\nu \mathcal{F}_{\nu\mu}) \\
 & + \tilde{\nabla}_\mu \left(\Upsilon_{(s)\text{new}}^\mu - \frac{1}{T} (\mathcal{U} - \mu \mathcal{V}) \bar{n}^\mu - c_1 u^{[\nu} \xi^{\mu]} n_\nu + c_2 \bar{n}^\mu \right) \geq 0,
 \end{aligned} \tag{5.20}$$

where we have used the bulk Euler relation $E + P = ST + \mu Q$, and defined

$$\begin{aligned}
 \mathcal{E}_\phi &= \tilde{\nabla}_\mu (\mathcal{F} \tilde{\xi}^\mu) - (n^\mu \xi_\mu) F - \mathcal{O}_\chi(f_1)^{\mu\nu} n_\mu \frac{1}{T} \partial_\nu T - \mathcal{O}_\chi(f_2)^{\mu\nu} n_\mu T \partial_\nu \nu \\
 & - \mathcal{O}_\chi(g_1)^\mu{}_\alpha n_\mu \epsilon^{\alpha\nu\rho\sigma} u_\nu \partial_\rho u_\sigma - \frac{1}{2} \mathcal{O}_\chi(g_2)^\mu{}_\alpha n_\mu \epsilon^{\alpha\nu\rho\sigma} u_\nu \mathcal{F}_{\rho\sigma}, \\
 \mathcal{E}_f &= \nabla_\mu \left(\mathcal{Y} n^\mu - (n^\nu \xi_\nu) \mathcal{F} \tilde{\xi}^\mu \right) - P - f_1 \xi^\nu \frac{1}{T} \partial_\nu T - f_2 \xi^\nu T \partial_\nu \nu \\
 & + g_1 \epsilon^{\mu\nu\rho\sigma} u_\mu \zeta_\nu \partial_\rho u_\sigma + g_2 \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\mu \zeta_\nu \mathcal{F}_{\rho\sigma}.
 \end{aligned} \tag{5.21}$$

The condition of positive semi-definiteness implies the surface thermodynamics

$$\begin{aligned}
 d(\mathcal{E} - \lambda f_1) &= T d \left(\mathcal{S} - \frac{\lambda}{T} (f_1 - \mu f_2) \right) + \mu d(\mathcal{Q} - \lambda f_2) - \frac{1}{2} \mathcal{F} d\tilde{\chi}, \\
 \mathcal{E} - \mathcal{Y} &= T \mathcal{S} + \mu \mathcal{Q},
 \end{aligned} \tag{5.22}$$

and the relations

$$\mathcal{U} = g_1, \quad \mathcal{V} = g_2, \tag{5.23}$$

which are exactly the same as the ones found using the equilibrium partition function. The second law also implies the corrections to the entropy current³⁵

$$\Upsilon_{(s)\text{new}}^\mu = \frac{1}{T} (\mathcal{U} - \mu \mathcal{V} - T c_2) \bar{n}^\mu + c_1 u^{[\nu} \xi^{\mu]} n_\nu. \tag{5.25}$$

After imposing all of these, the second law of thermodynamics will turn into

$$\frac{u^\mu \xi_\mu - \mu}{T} \mathcal{E}_\phi + \frac{u^\mu n_\mu}{T} \mathcal{E}_f \geq 0, \tag{5.26}$$

which will admit a general solution

$$u^\mu \xi_\mu - \mu = \alpha \mathcal{E}_\phi + (\beta + \beta') \mathcal{E}_f, \quad u^\mu n_\mu = (\beta - \beta') \mathcal{E}_\phi + \varsigma \mathcal{E}_f, \tag{5.27}$$

³⁵Note that, we can always modify the entropy currents as

$$J_{(b)\text{ent}}^\mu \rightarrow J_{(b)\text{ent}}^\mu + \nabla_\nu X^{[\mu\nu]}, \quad J_{(s)\text{ent}}^\mu \rightarrow J_{(b)\text{ent}}^\mu + n_\nu X^{[\mu\nu]}, \tag{5.24}$$

without changing the second law, hence the entropy currents always have this ambiguity. Interestingly, using this ambiguity we can get rid of both the c_1 and c_2 contributions from the theory.

with $\varsigma \geq 0$, $\alpha\varsigma \geq \beta^2$ and an arbitrary β' . These are the respective Josephson equation and the equation of motion for f outside equilibrium which determines $u^\mu \xi_\mu$ and $u^\mu n_\mu$ respectively. On the other hand, the second law of thermodynamics leaves $n^\mu \xi_\mu$ undetermined. In equilibrium $u^\nu \xi_\nu = \mu$ and $u^\mu n_\mu = 0$, which implies the equilibrium versions of the Josephson and Young-Laplace equation respectively

$$\mathcal{E}_\phi = 0, \quad \mathcal{E}_f = 0, \quad (5.28)$$

which are same as the ones derived using an equilibrium partition function. It is worthwhile noting that outside equilibrium, contrary to the ordinary fluid case discussed in the previous section, the equation of motion of f is not the Young-Laplace equation.

5.1.3 Surface entropy current for 2+1 dimensional superfluids

In this subsection we will give the entropy current analysis for 2+1 dimensional superfluids with a surface. We will only focus on the boundary computation here, for simplicity. As pointed out in the previous section, the only way in which the bulk interacts with the boundary in the second law (5.19), is via the bulk entropy current correction $\Upsilon_{(b)\text{new}}^\mu$. In 2 + 1 dimensions, the form of $\Upsilon_{(b)\text{new}}^\mu$ is same as in the 3 + 1 dimensional case in the parity-even sector, but is quite different in the parity-sector. It is given by³⁶

$$\begin{aligned} \Upsilon_{(b)\text{new}}^\mu = & \nabla_\nu \left(c_1 u^{[\mu} \xi^{\nu]} + c_2 \epsilon^{\mu\nu\rho} u_\rho + c_3 \epsilon^{\mu\nu\rho} \xi_\rho \right) \\ & + \left(\frac{f_1}{T} 2u^{[\mu} \xi^{\nu]} \frac{1}{T} \partial_\nu T + \frac{u^\mu \xi_\mu - \mu}{T} \mathcal{O}_\chi (f_1)^{\mu\nu} \frac{1}{T} \partial_\nu T \right) \\ & + \left(\frac{f_2}{T} 2u^{[\mu} \xi^{\nu]} T \partial_\nu \nu + \frac{u^\mu \xi_\mu - \mu}{T} \mathcal{O}_\chi (f_2)^{\mu\nu} T \partial_\nu \nu \right) \\ & - \left(\frac{\kappa_1}{T} \epsilon^{\mu\nu\rho} \frac{1}{T} \partial_\nu (T u_\rho) - \frac{\kappa_1}{T} \frac{\partial(m_\omega - m_{B\nu})}{\partial\chi} \zeta^\mu \epsilon^{\alpha\nu\rho} u_\alpha \partial_\nu u_\rho \right) \\ & - \left(\frac{\kappa_2}{T} \epsilon^{\mu\nu\rho} \left(\frac{1}{2} \mathcal{F}_{\nu\rho} + u_\rho T \partial_\nu \nu \right) - \frac{u^\mu \xi_\mu - \mu}{T} \frac{\partial\kappa_2}{\partial\chi} \zeta^\mu \frac{1}{2} \epsilon^{\alpha\nu\rho} u_\alpha \mathcal{F}_{\nu\rho} \right) \\ & + \left(\frac{\kappa_3}{T} \epsilon^{\mu\nu\rho} \zeta_\nu \frac{1}{T} \partial_\rho T + \frac{u^\mu \xi_\mu - \mu}{T} \mathcal{O}_\chi (\kappa_3)^\mu_\nu \epsilon^{\nu\rho\sigma} u_\rho \frac{1}{T} \partial_\sigma T \right) \\ & + \left(\frac{\kappa_4}{T} \epsilon^{\mu\nu\rho} \zeta_\nu T \partial_\rho \nu + \frac{u^\mu \xi_\mu - \mu}{T} \mathcal{O}_\chi (\kappa_4)^\mu_\nu \epsilon^{\nu\rho\sigma} u_\rho T \partial_\sigma \nu \right). \end{aligned} \quad (5.29)$$

On the other hand, the most generic surface currents are given as³⁷

$$\begin{aligned} T_{(s)}^{\mu\nu} &= (\mathcal{E} - \mathcal{Y}) u^\mu u^\nu - \mathcal{Y} (\mathcal{G}^{\mu\nu} - n^\mu n^\nu) + \mathcal{F} \tilde{\xi}^\mu \tilde{\xi}^\nu + 2\mathcal{U} u^{(\mu} \tilde{n}^{\nu)} + \Pi_{(s)}^{\mu\nu}, \\ J_{(s)}^\mu &= \mathcal{Q} u^\mu - \mathcal{F} \tilde{\xi}^\mu + \mathcal{V} \tilde{n}^\mu + \Upsilon_{(s)}^\mu, \\ J_{(s)\text{ent}}^\mu &= \mathcal{S} u^\mu + \Upsilon_{(s)\text{ent}}^\mu, \quad \Upsilon_{(s)\text{ent}}^\mu = -\frac{u_\nu}{T} \Pi_{(s)}^{\mu\nu} - \frac{\mu}{T} \Upsilon_{(s)}^\mu + \Upsilon_{(s)\text{new}}^\mu, \end{aligned} \quad (5.30)$$

³⁶We do not know of any reference which discusses generic first order corrections to entropy current for 2 + 1 dimensional superfluids. However, we can use the results of [27] to work out the generic $\Upsilon_{(b)\text{new}}^\mu$ (see footnote 33).

³⁷As in the previous examples, we have not considered contributions proportional to n^μ in the surface currents for clarity of presentation.

where $\bar{n}^\mu = \epsilon^{\mu\nu\rho} u_\nu n_\rho$. It should be noted that in $2+1$ dimensions, \bar{n}^μ can be written in terms of u^μ , n^μ and $\tilde{\xi}^\mu$, but we keep it in this format in hindsight. Up to first order in the bulk and ideal order at the boundary, the second law (5.19) takes the form

$$\begin{aligned} & \tilde{\nabla}_\mu J_{(s)\text{ent}}^\mu - n_\mu J_{(b)\text{ent}}^\mu = \\ & -\frac{u^\mu}{T} \left(\partial_\mu \mathcal{E} - T \partial_\mu \mathcal{S} - \mu \partial_\mu \mathcal{Q} + \mathcal{F} (\mu \partial_\mu \mu - \bar{\lambda} \partial_\mu \bar{\lambda}) - (\lambda f_1 - \bar{\lambda} \kappa_3) \frac{1}{T} \partial_\mu T - (\lambda f_2 - \bar{\lambda} \kappa_4) T \partial_\mu \nu \right) \\ & - \frac{1}{T} (\mathcal{E} - \mathcal{Y} + T \mathcal{S} + \mu \mathcal{Q}) \tilde{\nabla}_\mu u^\mu + \frac{u^\mu \xi_\mu - \mu}{T} \mathcal{E}_\phi + \frac{u^\mu n_\mu}{T} \mathcal{E}_f \\ & - 2 (\mathcal{U} - \kappa_1) u^{(\mu} \bar{n}^{\nu)} \nabla_\mu \left(\frac{u_\nu}{T} \right) - \frac{1}{T} (\mathcal{V} - \kappa_2) \bar{n}^\mu (T \nabla_\mu \nu + u^\nu \mathcal{F}_{\nu\mu}) \\ & + \tilde{\nabla}_\mu \left(\Upsilon_{(s)\text{new}}^\mu - \frac{1}{T} \mathcal{U}_\xi \bar{n}^\mu + \nu \mathcal{V}_\xi \bar{n}^\mu + c_1 u^{[\mu} \xi^{\nu]} n_\nu + c_2 \epsilon^{\mu\nu\rho} n_\nu u_\rho + c_3 \epsilon^{\mu\nu\rho} n_\nu \xi_\rho \right) \geq 0, \end{aligned} \quad (5.31)$$

where, we have used the bulk Euler relation $E + P = TS + \mu Q$, and defined

$$\begin{aligned} \mathcal{E}_\phi &= \tilde{\nabla}_\mu (\mathcal{F} \tilde{\xi}^\mu) - (n^\mu \xi_\mu) F - \mathcal{O}_\chi (f_1)^{\mu\nu} n_\mu \frac{1}{T} \partial_\nu T - \mathcal{O}_\chi (f_2)^{\mu\nu} n_\mu T \partial_\nu \nu \\ & - \frac{\partial \kappa_1}{\partial \chi} \zeta^\mu n_\mu \epsilon^{\alpha\nu\rho} u_\alpha \partial_\nu u_\rho - \frac{\partial \kappa_2}{\partial \chi} \zeta^\mu n_\mu \frac{1}{2} \epsilon^{\alpha\nu\rho} u_\alpha \mathcal{F}_{\nu\rho} \\ & - \mathcal{O}_\chi (\kappa_3)^\mu_\nu n_\mu \epsilon^{\nu\rho\sigma} u_\rho \frac{1}{T} \partial_\sigma T - \mathcal{O}_\chi (\kappa_4)^\mu_\nu n_\mu \epsilon^{\nu\rho\sigma} u_\rho T \partial_\sigma \nu, \\ \mathcal{E}_f &= \nabla_\mu \left(\mathcal{Y} n^\mu - (n^\nu \xi_\nu) \mathcal{F} \tilde{\xi}^\mu \right) - P - f_1 \xi^\nu \frac{1}{T} \partial_\nu T - f_2 \xi^\nu T \partial_\nu \nu - \kappa_2 \epsilon^{\mu\nu\rho} u_\mu \frac{1}{2} \mathcal{F}_{\nu\rho} \\ & - \kappa_1 \epsilon^{\mu\nu\rho} u_\mu \partial_\nu u_\rho - \kappa_3 \epsilon^{\mu\nu\rho} \zeta_\mu u_\nu \frac{1}{T} \partial_\rho T - \kappa_4 \epsilon^{\mu\nu\rho} \zeta_\mu u_\nu T \partial_\rho \nu. \end{aligned} \quad (5.32)$$

Demanding positive definiteness, we can read out the surface thermodynamics

$$\begin{aligned} d\mathcal{C} &= - \left(\mathcal{S} - \frac{\lambda}{T} (f_1 - \mu f_2) + \frac{\bar{\lambda}}{T} (\kappa_3 - \mu \kappa_4) \right) dT \\ & - d \left(\mathcal{Q} - \lambda f_2 + \bar{\lambda} \kappa_4 \right) d\mu - \mathcal{F} (\mu d\mu - \bar{\lambda} d\bar{\lambda}), \\ \mathcal{E} - \mathcal{Y} &= T \mathcal{S} + \mu \mathcal{Q}. \end{aligned} \quad (5.33)$$

and the constraints

$$\mathcal{U} = \kappa_1, \quad \mathcal{V} = \kappa_2, \quad (5.34)$$

which are exactly the same as found using the equilibrium partition function. The respective first law of thermodynamics has been discussed in appendix A. Furthermore, we get the correction to the entropy current³⁸

$$\Upsilon_{(s)\text{new}}^\mu = \frac{1}{T} (\mathcal{U} - \mu \mathcal{V}) \bar{n}^\mu - c_1 u^{[\mu} \xi^{\nu]} n_\nu - c_2 \epsilon^{\mu\nu\rho} n_\nu u_\rho - c_3 \epsilon^{\mu\nu\rho} n_\nu \xi_\rho. \quad (5.35)$$

After implementing all of these constraints, the second law takes the form

$$\frac{u^\mu \xi_\mu - \mu}{T} \mathcal{E}_\phi + \frac{u^\mu n_\mu}{T} \mathcal{E}_f \geq 0, \quad (5.36)$$

³⁸We can remove the c_1 , c_2 , c_3 dependence of the system, by using the entropy current ambiguity (see footnote 35).

which can be solved, just like in the $3 + 1$ dimensional case, by

$$u^\mu \xi_\mu - \mu = \alpha \mathcal{E}_\phi + (\beta + \beta') \mathcal{E}_f, \quad u^\mu n_\mu = (\beta - \beta') \mathcal{E}_\phi + \varsigma \mathcal{E}_f, \quad (5.37)$$

with $\varsigma \geq 0$, $\alpha \varsigma \geq \beta^2$ and an arbitrary β' . These are the respective Josephson equation and equation of motion for f outside equilibrium which determines $u^\mu \xi_\mu$ and $u^\mu n_\mu$ respectively. Again, the second law of thermodynamics leaves $n^\mu \xi_\mu$ undetermined. In equilibrium, we recover the equilibrium version of the Josephson and Young-Laplace equations respectively

$$\mathcal{E}_\phi = 0, \quad \mathcal{E}_f = 0, \quad (5.38)$$

which are same as the ones derived using an equilibrium partition function.

5.2 Ripples on the surface

After studying the structure of the leading order surface equations away from equilibrium, in this section we shall study the nature of linearized fluctuations about an equilibrium configuration. For simplicity, we shall confine ourselves to the discussion in $2+1$ dimensions.

Ordinary fluids. Let us first consider the case of ordinary charged fluids in flat spacetime. We choose the coordinates $\{x^\mu\} = \{t, x, y\}$ with the flat Minkowski metric $\eta = \text{diag}\{-1, 1, 1\}$. We will consider one of the simplest equilibrium configurations, where the ordinary fluid fills the upper half spacetime $y \geq 0$, so that the equilibrium fluid variables take the form

$$T(t, x, y) = T_0, \quad u^\mu(t, x, y) = (1, 0, 0), \quad f(t, x, y) = y. \quad (5.39)$$

The line $f = y = 0$ is the fluid surface. For such a configuration to exist, the equilibrium pressure must be uniform everywhere. Also, since the extrinsic curvature of the line vanishes, this uniform equilibrium pressure must vanish as well $P(T_0) = 0$.³⁹ Note that although the equilibrium pressure vanishes everywhere, the entropy density $S(T_0) = P'(T_0)$ and the energy density $E(T_0) = T_0 S(T_0) - P(T_0) = T_0 S(T_0)$ remains uniformly non-zero. Now, let us consider linearized fluctuations about this configuration

$$T = T_0 + \epsilon \delta T + \mathcal{O}(\epsilon^2), \quad u^\mu = (1, \epsilon \delta u^x, \epsilon \delta u^y) + \mathcal{O}(\epsilon^2), \quad f = y + \delta f + \mathcal{O}(\epsilon^2). \quad (5.40)$$

Note that in (5.40), u^μ remains unit normalized up to the relevant order, i.e. $u^\mu u_\mu = -1 + \mathcal{O}(\epsilon^2)$. The linearized equations in the bulk, which follow from the

³⁹Note that the vanishing of the extrinsic curvature only implies that the pressure difference at the surface vanishes. If we consider a scenario similar to the one in [33], where a plasma fluid is separated from the vacuum by a surface, then the surface pressure and hence the equilibrium pressure everywhere in the bulk for the configuration (5.39) must vanish. This may be achieved if the equation of state is of the form $P(T) = A T^\alpha - B$. In such system, the configuration (5.39) can exist as a metastable state at the phase transition temperature T_0 .

conservation of the leading order energy-momentum tensor in (5.2), are given by,

$$S(T_0) (\partial_x \delta u^x + \partial_y \delta u^y) + S'(T_0) \partial_t \delta T = 0, \quad (5.41a)$$

$$S(T_0) \left(\partial_t \delta u^x + \frac{1}{T_0} \partial_x \delta T \right) = 0, \quad (5.41b)$$

$$S(T_0) \left(\partial_t \delta u^y + \frac{1}{T_0} \partial_y \delta T \right) = 0. \quad (5.41c)$$

As we have argued in 5.1.1, $n_\mu u^\mu$ at leading order must vanish due to the second law, i.e. $\gamma = 0$. This serves as the additional equation required for determining the additional variable at the surface. In the linearized approximation this equation is given by

$$\partial_t \delta f = -\delta u^y. \quad (5.42)$$

Using this and the leading order surface energy-momentum tensor (5.3), the surface conservation laws take the form

$$\mathcal{S}(T_0) \partial_x \delta u^x + \mathcal{S}'(T_0) \partial_t \delta T = 0, \quad (5.43a)$$

$$\mathcal{S}(T_0) \left(\partial_t \delta u^x + \frac{1}{T_0} \partial_x \delta T \right) = 0, \quad (5.43b)$$

$$\mathcal{E}(T_0) \partial_t^2 \delta f - \mathcal{Y}(T_0) \partial_x^2 \delta f = S(T_0) \delta T. \quad (5.43c)$$

Now, the procedure for solving these equations as outlined in section 1 includes first solving the 4 surface equations (5.42), (5.43) for δu^x , δu^y , δT and δf at the surface, and then use the solutions as a boundary condition for solving the remaining 3 bulk equations (5.41) for δu^x , δu^y and δT . The boundary condition should be specified at $f = 0$. In the linearized approximation that we are working in, it suffices to impose the boundary condition at $y = 0$.

In the classical computation of capillary waves [34], the surface entropy \mathcal{S} is considered to be zero, or equivalently, a constant surface tension is assumed. In this limit, (5.43a) and (5.43b) are automatically satisfied. This implies that the set of allowed boundary conditions is less constrained compared to the more general case. Thus, the bulk equations, in that case, may be solved with partially arbitrary boundary conditions, as long as (5.43c) and (5.42) are ensured to be satisfied.

In order to obtain the dispersion relation of capillary waves, in the absence of any external gravitational field, the equations (5.41b), (5.41c), (5.42) and (5.43c) are solved by

$$\begin{aligned} \delta u^x(t, x, y) &= -\delta f_0 \frac{k_x \omega}{\kappa} \cos(k_x x + \omega t) e^{-\kappa y}, & \delta u^y(t, x, y) &= \delta f_0 \omega \sin(k_x x + \omega t) e^{-\kappa y}, \\ \delta T(t, x, y) &= \delta f_0 \frac{T_0 \omega^2}{\kappa} \cos(k_x x + \omega t) e^{-\kappa y}, & \delta f(t, x) &= \delta f_0 \cos(k_x x + \omega t), \\ \text{with } \omega &= \pm k_x \sqrt{\frac{\kappa \mathcal{Y}}{E + \kappa \mathcal{Y}}}, \end{aligned} \quad (5.44)$$

where δf_0 is the wave amplitude, k_x is the wavenumber and ω is the wave frequency of the linearized fluctuation. The remaining equation (5.41a) provides a condition for determining the damping factor $\kappa \geq 0$

$$\kappa (\kappa^2 - k_x^2) + T_0 \frac{E'}{E} (\kappa - |k_x|) k_x^2 + \frac{E}{\mathcal{Y}} (\kappa^2 - k_x^2) + T_0 \frac{E'}{E} |k_x|^3 = 0. \quad (5.45)$$

For small $|k_x|$, this condition simply sets $\kappa = |k_x|$, which implies the well-known dispersion relation of the form $\omega \approx \pm k_x^{3/2} \sqrt{\mathcal{Y}/E}$.

However, if we take into account a non-zero surface entropy, then the boundary conditions for solving the bulk equations must satisfy all the equations in (5.43) and (5.42). This completely determines the possible set of boundary conditions. In fact, (5.43) and (5.42) admits a sinusoidal solution with the following dispersion relations

$$\omega = \pm k \sqrt{\frac{\mathcal{Y}}{\mathcal{E}}}, \quad \omega = \pm k \sqrt{\frac{\mathcal{S}}{\partial \mathcal{E} / \partial T}}, \quad (5.46)$$

We see that there are two sound-like modes on the surface. We can solve the bulk equations (5.41) with the sound modes as the boundary condition at $y = 0$. For instance, the full bulk solution corresponding to the first dispersion relation in (5.46) takes the form

$$\begin{aligned} \delta u^x(t, x, y) &= \delta f_0 \frac{k_x \omega}{\kappa} \sin(\kappa y) \cos(k_x x + \omega t), \\ \delta u^y(t, x, y) &= \delta f_0 \omega \cos(\kappa y) \sin(k_x x + \omega t), \\ \delta T(t, x, y) &= -\delta f_0 \frac{\omega^2 T_0}{\kappa} \sin(\kappa y) \cos(k_x x + \omega t), \\ \delta f(t, x) &= \delta f_0 \cos(k_x x + \omega t), \end{aligned} \quad (5.47)$$

where, $\kappa = |k_x| \sqrt{\frac{E'(T_0)}{P'(T_0)} \frac{\mathcal{Y}(T_0)}{\mathcal{E}(T_0)} - 1}$ and the dispersion is, $\omega = \pm k_x \sqrt{\frac{\mathcal{Y}(T_0)}{\mathcal{E}(T_0)}}$.

It can be easily checked that (5.47) solves both the bulk (5.41) and surface equations (5.43) simultaneously. There also exists a similar sinusoidal solution corresponding to the second dispersion in (5.46).

Note that it should be possible to have both, the capillary waves in (5.44), as well as the tiny ripples (5.47) on the surface of the same fluid. If the amplitude of the waves is large compared to the thickness of the surface, then neglecting the surface entropy would be a legitimate approximation. Hence, in that case, we shall have capillary waves as in (5.44). On the other hand, if the amplitude of the surface waves is small or comparable to the surface thickness, then waves like (5.47) would be generated.⁴⁰

Superfluids. We now move on to surface linear fluctuations in a $2 + 1$ dimensional superfluid. To start with, we will consider an equilibrium configuration similar to (5.39), with the superfluid phase filling half spacetime $y \geq 0$

$$\begin{aligned} T(t, x, y) &= T_0, & \mu(t, x, y) &= \mu_0, & u^\mu(t, x, y) &= (1, 0, 0), & f(t, x, y) &= y, \\ \phi(t, x, y) &= \phi_0, & \xi^\mu(t, x, y) &= (-\mu_0, 0, 0), & \chi(t, x, y) &= \mu_0^2, & \lambda = \bar{\lambda} &= 0. \end{aligned} \quad (5.48)$$

⁴⁰In this sense, the linearized solution (5.47) is similar to the third sound mode on superfluid surfaces [35].

We consider the following linearized fluctuations about this equilibrium configuration

$$\begin{aligned}
 T &= T_0 + \epsilon \delta T + \mathcal{O}(\epsilon^2), & \mu &= \mu_0 - \epsilon \partial_t \delta \phi + \mathcal{O}(\epsilon^2), & u^\mu &= (1, \epsilon \delta u^x, \epsilon \delta u^y) + \mathcal{O}(\epsilon^2), \\
 \phi &= \phi_0 + \epsilon \delta \phi + \mathcal{O}(\epsilon^2), & \xi^\mu &= \{-\mu_0 + \epsilon \partial_t \delta \phi, -\epsilon \partial_x \delta \phi, -\epsilon \partial_y \delta \phi\} + \mathcal{O}(\epsilon^2), \\
 f &= y + \epsilon \delta f + \mathcal{O}(\epsilon^2), & \chi &= \mu_0^2 - 2\epsilon \mu_0 \partial_t \delta \phi + \mathcal{O}(\epsilon^2), \\
 \lambda &= \epsilon (\partial_y \delta \phi - \mu_0 \delta u^y) + \mathcal{O}(\epsilon^2), & \bar{\lambda} &= -\epsilon (\partial_x \delta \phi - \mu_0 \delta u^x) + \mathcal{O}(\epsilon^2),
 \end{aligned} \tag{5.49}$$

where we have used $u^\mu n_\mu = 0$ at ideal order. Note that the curl free condition for the superfluid velocity and the Josephson condition $u^\mu \xi_\mu = \mu$ have already been implemented in the ansatz (5.49), up to the relevant order. The surface equations are given by conservation of currents in (3.10), which includes parity-odd effects.

With the most general analysis of the fluctuation equations, we found that a system with a generic equation of state $\mathcal{Y} = \mathcal{Y}(T, \mu, \lambda, \bar{\lambda})$, exhibits 6 independent modes at the surface. These modes can further be used as boundary conditions to solve the bulk equations. For simplicity, however, here we consider a simplified equation of state

$$\mathcal{Y} = T\mathcal{Y}_1 + \bar{\lambda}\mathcal{Y}_2, \tag{5.50}$$

where $\mathcal{Y}_1, \mathcal{Y}_2$ are constants. With this ansatz, the linearized surface conservation equations following from the leading order currents (5.14) and (3.10), together with the condition $\gamma = 0$, yield

$$\begin{aligned}
 T_0 \mathcal{Y}_1 \partial_x \delta u^x + \mu_0 \mathcal{Y}_2 \partial_t \delta u^x &= \mu_0 F (\partial_y \delta \phi + \mu_0 \partial_t \delta f), \\
 -\mathcal{Y}_1 (\partial_x \delta T + T_0 \partial_t \delta u^x) - \mathcal{Y}_2 (2\mu_0 \partial_x \delta u^x - \partial_t^2 \delta \phi) &= 0, \\
 \mathcal{Y}_1 T_0 \partial_x^2 \delta f + 2\mu_0 \mathcal{Y}_2 \partial_t \partial_x \delta f &= -S \delta T + (Q + \mu_0 F) \partial_t \delta \phi, \\
 \mathcal{Y}_2 \partial_t \delta u^x &= F_0 (\partial_y \delta \phi + \mu_0 \partial_t \delta f), \\
 \partial_t \delta f + \delta u^y &= 0.
 \end{aligned} \tag{5.51}$$

This system of equations exhibits a sinusoidal solution of the form $e^{i(\omega t - k_x x)} e^{-\ell y}$, only if ω satisfies

$$\mu_0 \omega^2 (S \mathcal{Y}_2 \omega - \mathcal{Y}_1 (Q + \mu_0 F) k_x) - \ell k_x^2 \mathcal{Y}_1 (2\mu_0 \mathcal{Y}_2 \omega + T_0 \mathcal{Y}_1 k_x) = 0, \tag{5.52}$$

where k_x is the x -momentum, and ω is the frequency. This equation leads to 3 modes $\omega \propto k$ (implying that 3 out of 6 modes were lifted due to the specific choice of the equation of state⁴¹), one of which is a sound-like mode. We observe that none of these three modes come with a parity conjugate $k \rightarrow -k$, which can be seen as an imprint of parity-odd effects on the spectrum of linearized fluctuations.

One quick check which one can perform for this phenomenon is by taking $\ell = 0$. In this limit, equation (5.52) boils down to $\omega^2 (S \mathcal{Y}_2 \omega - \mathcal{Y}_1 (Q + \mu_0 F) k_x) = 0$, which implies dispersion relations $\omega^2 = 0$ and $\omega = \frac{\mathcal{Y}_1 (Q + \mu_0 F)}{\mathcal{Y}_2 S} k_x$. Though the first solution respects parity in this limit, the second clearly breaks it, as is expected for a system with no parity invariance.

⁴¹For instance, in ordinary fluids, the sound modes disappear if we consider an equation of state where the pressure is linear in temperature. This is because the velocity of sound becomes infinite in this limit.

6 Discussion

In this paper, we have worked out the leading order surface energy-momentum tensor and charge current for a finite bubble of superfluid, both in equilibrium and slightly away from it. In equilibrium, we were able to write down the most general Euclidean effective action for the Goldstone boson and the shape-field (in one lower dimension), coupled to arbitrary slowly varying background fields. By appropriately varying this action, we obtained all surface currents. Away from equilibrium, we used the second law of thermodynamics, implemented via an entropy current with a positive semi-definite divergence. Our near equilibrium results reduce to those obtained from the effective action, upon restricting to the stationary sector.

The ideal order surface currents contain new terms, compared to their bulk counterparts, which are entirely determined by the first order bulk transport coefficients. This exercise has revealed new parity-even and parity-odd terms in the ideal order surface currents. In the case of the parity-odd terms, we have shown that they leave an imprint in the spectrum of linearized fluctuations. Such terms are also present in the surface currents of Galilean superfluids, which we have obtained by a null-reduction of $4 + 1$ dimensional null superfluids. Hence, such new effects should also be relevant in realistic non-relativistic situations.

The parity-odd surface effects that we discussed here are relevant for theories with microscopic parity violation,⁴² but they may also be present as an emergent parity odd phenomenon. In order to better understand the nature of the physical systems in which our results would play an important role, it would be interesting to write down Kubo-like formulae for the first order parity-odd superfluid coefficients, along the lines of [37].

The results found here are extremely relevant in the context of black holes via the AdS/CFT correspondence. In this holographic context, the space-filling configurations of the boundary fluid have a one-to-one correspondence with slowly varying black brane configurations in the bulk [38]. It is also possible to generalize such maps to the context where the plasma of the deconfined phase fills the space partially while the rest of space is occupied by the confined phase [33, 39]. In the large N limit, such situations may be described by a plasma fluid separated from the vacuum by a surface in the hydrodynamic approximation. The holographic dual of such fluid configurations is a combination of black branes and the AdS-soliton patched up in a suitable fashion to account for the fluid surface at the boundary [7, 40–42]. Similarly, the holographic dual of the space filling superfluid phase are AdS hairy black holes [15]. It would be extremely interesting to construct the holographic duals of the superfluid bubbles discussed in this paper, along the lines of [33]. Such hairy black holes, besides being new and interesting solutions of the Einstein equations, may provide a suitable microscopic setting for a better understanding of the functional dependence of the surface tension on its arguments.

⁴²For instance, in theories with anomalies, there may be additional terms in the first order bulk currents that are entirely determined by the anomaly coefficient [2, 3, 36]. Although we have refrained from discussing such (non-gauge invariant) terms in this paper, we hope to return to this question in a later work.

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A Surface thermodynamics in 2+1 dimensions

In section 3 we derived the surface Euler relation and Gibbs-Duhem relation for a superfluid bubble, which take the form

$$\begin{aligned}
 -d\mathcal{Y} &= \left(\mathcal{S} - \frac{\lambda}{T}(f_1 - \mu f_2) + \frac{\bar{\lambda}}{T}(\kappa_3 - \mu \kappa_4) \right) dT + \left(\mathcal{Q} - \lambda f_2 + \bar{\lambda} \kappa_4 \right) d\mu + \mathcal{F}(\mu d\mu - \bar{\lambda} d\bar{\lambda}), \\
 \mathcal{E} - \mathcal{Y} &= T\mathcal{S} + \mu\mathcal{Q}.
 \end{aligned}
 \tag{A.1}$$

Though these relations are correct, they mix parity-even and parity-odd sectors. In this appendix, we will write down mutually independent thermodynamics for parity-even and parity-odd sectors on the surface, and derive the respective first law of thermodynamics.

The thermodynamic ensemble, as used in (A.1), is described by $\mathcal{Y}(T, \mu, \bar{\lambda})$. Using the fact that $\bar{\lambda}^2 = \mu^2 - \tilde{\chi}$, and performing a Taylor expansion in $\bar{\lambda}$, we can split this ensemble into a parity-even and a parity-odd sector

$$\mathcal{Y}(T, \mu, \bar{\lambda}) = \mathcal{Y}_+(T, \mu, \tilde{\chi}) + \bar{\lambda} \mathcal{Y}_-(T, \mu, \tilde{\chi}).
 \tag{A.2}$$

Note that both the functions \mathcal{Y}_+ and \mathcal{Y}_- are purely parity-even. Keeping in mind the inflow from the bulk, we define mutually independent Gibbs-Duhem and Euler relations in the parity-even and parity-odd sectors, in terms of \mathcal{Y}_+ and \mathcal{Y}_- respectively

$$\begin{aligned}
 -d\mathcal{Y}_+ &= \left(\mathcal{S}_+ - \frac{\lambda}{T}(f_1 - \mu f_2) \right) dT + \left(\mathcal{Q}_+ - \lambda f_2 \right) d\mu + \frac{1}{2} \mathcal{F}_+ d\tilde{\chi}, & \mathcal{E}_+ - \mathcal{Y}_+ &= T\mathcal{S}_+ + \mu\mathcal{Q}_+, \\
 -d\mathcal{Y}_- &= \left(\mathcal{S}_- + \frac{1}{T}(\kappa_3 - \mu \kappa_4) \right) dT + \left(\mathcal{Q}_- + \kappa_4 \right) d\mu + \frac{1}{2} \mathcal{F}_- d\tilde{\chi}, & \mathcal{E}_- - \mathcal{Y}_- &= T\mathcal{S}_- + \mu\mathcal{Q}_-.
 \end{aligned}
 \tag{A.3}$$

Comparing these to the parity-mixed expressions, we can read out the parity splitting of energy, charge, entropy and superfluid density respectively

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_+ + \bar{\lambda} \left(\mathcal{E}_- - \frac{\mu^2}{\mu^2 - \tilde{\chi}} \mathcal{Y}_- \right), & \mathcal{Q} &= \mathcal{Q}_+ + \bar{\lambda} \left(\mathcal{Q}_- - \frac{\mu}{\mu^2 - \tilde{\chi}} \mathcal{Y}_- \right), \\ \mathcal{S} &= \mathcal{S}_+ + \bar{\lambda} \mathcal{S}_-, & \mathcal{F} &= \mathcal{F}_+ + \bar{\lambda} \left(\mathcal{F}_- + \frac{1}{\mu^2 - \tilde{\chi}} \mathcal{Y}_- \right).\end{aligned}\quad (\text{A.4})$$

Using (A.3), it is easy to derive the first law of thermodynamics for parity-even and parity-odd sectors respectively

$$\begin{aligned}d(\mathcal{E}_+ - \lambda f_1) &= T d\left(\mathcal{S}_+ - \frac{\lambda}{T}(f_1 - \mu f_2)\right) + \mu d(\mathcal{Q}_+ - \lambda f_2) - \frac{1}{2} \mathcal{F}_+ d\tilde{\chi}, \\ d(\mathcal{E}_- + \kappa_3) &= T d\left(\mathcal{S}_- + \frac{1}{T}(\kappa_3 - \mu \kappa_4)\right) + \mu d(\mathcal{Q}_- + \kappa_4) - \frac{1}{2} \mathcal{F}_- d\tilde{\chi}.\end{aligned}\quad (\text{A.5})$$

B Equation of motion for the shape-field and the Young-Laplace equation

In this appendix, we rigorously show that in the stationary case, the Young-Laplace equation that follows by projecting the surface conservation equation along n_μ , is identical to the equation of motion of f which follows from the equilibrium partition function, up to all orders in derivatives. Let us start with the most generic partition function variation parametrized as

$$\begin{aligned}\delta W &= \int d^4x \sqrt{-\mathcal{G}} \theta(f) \left(\frac{1}{2} T_{(b)}^{\mu\nu} \delta \mathcal{G}_{\mu\nu} + J_{(b)}^\mu \delta \mathcal{A}_\mu \right) \\ &+ \int d^4x \sqrt{-\mathcal{G}} \theta(-f) \left(\frac{1}{2} T_{(e)}^{\mu\nu} \delta \mathcal{G}_{\mu\nu} + J_{(e)}^\mu \delta \mathcal{A}_\mu \right) \\ &+ \int d^4x \sqrt{-\mathcal{G}} \tilde{\delta}(f) \left(\frac{1}{2} T_{(s)}^{\mu\nu} \delta \mathcal{G}_{\mu\nu} + J_{(s)}^\mu \delta \mathcal{A}_\mu + \frac{Y}{\sqrt{\nabla_\mu f \nabla^\mu f}} \delta f \right),\end{aligned}\quad (\text{B.1})$$

where

$$\tilde{\delta}^{(n)}(f) = (-)^{n+1} (n^\mu \partial_\mu)^{n+1} \theta(f). \quad (\text{B.2})$$

The Young-Laplace seen as equation of motion of f is just $Y = 0$. On the other hand, we know that W is a gauge invariant scalar, so it must be invariant under a diffeomorphism and gauge variation of the constituent fields, parametrized by $\mathcal{X} = \{\vartheta^\mu, \Lambda_\vartheta\}$

$$\begin{aligned}\delta_{\mathcal{X}} \mathcal{G}_{\mu\nu} &= \nabla_\mu \vartheta_\nu + \nabla_\nu \vartheta_\mu, & \delta_{\mathcal{X}} \mathcal{A}_\mu &= \nabla_\mu (\Lambda_\vartheta + \mathcal{A}_\mu \vartheta^\mu) + \vartheta^\nu \mathcal{F}_{\nu\mu}, \\ \delta_{\mathcal{X}} f &= \vartheta^\mu \partial_\mu f = -\sqrt{\nabla_\mu f \nabla^\mu f} \vartheta^\mu n_\mu.\end{aligned}\quad (\text{B.3})$$

This leads to a set of identities

$$\begin{aligned}\theta(f) \left[\nabla_\mu T_{(b)}^{\mu\nu} - \mathcal{F}^{\nu\rho} J_{\rho}^{(b)} \right], & \quad \theta(f) \left[\nabla_\mu J_{(b)}^\mu \right] = 0, \\ \theta(f) \left[\nabla_\mu T_{(e)}^{\mu\nu} - \mathcal{F}^{\nu\rho} J_{\rho}^{(e)} \right], & \quad \theta(f) \left[\nabla_\mu J_{(e)}^\mu \right] = 0,\end{aligned}$$

$$\begin{aligned}
 \tilde{\delta}(f) \left[\nabla_\mu T_{(s)}^{\mu\nu} - \mathcal{F}^{\nu\rho} J_\rho^{(s)} - n_\mu (T_{(b)}^{\mu\nu} - T_{(e)}^{\mu\nu}) + n^\nu Y \right] &= 0, \\
 \tilde{\delta}(f) \left[\nabla_\mu J_{(s)}^\mu - n_\mu (J_{(b)}^\mu - J_{(e)}^\mu) \right] &= 0, \\
 \tilde{\delta}'(f) \left[T_{(s)}^{\mu\nu} n_\nu \right] &= 0, \quad \tilde{\delta}'(f) \left[J_{(s)}^\mu n_\mu \right] = 0. \quad (\text{B.4})
 \end{aligned}$$

From here it is clear that an analogous representation of the Young-Laplace equation $Y = 0$ is the n_μ component of the surface energy-momentum conservation equation

$$n_\nu \nabla_\mu T_{(s)}^{\mu\nu} = n_\nu \left(\mathcal{F}^{\nu\rho} J_\rho^{(s)} + n_\mu (T_{(b)}^{\mu\nu} - T_{(e)}^{\mu\nu}) \right). \quad (\text{B.5})$$

This equation can be thought of as extremizing the partition function W under a restricted variation where we Lie drag $\mathcal{G}_{\mu\nu}$ and \mathcal{A}_μ along $\vartheta^\mu = \vartheta n^\mu$ keeping f fixed. It might be beneficial to see this explicitly. Let the partition function have the form

$$W = \int d^4x \sqrt{-\mathcal{G}} \left[\theta(f) \mathcal{L}_{(b)} + \theta(-f) \mathcal{L}_{(e)} + \tilde{\delta}(f) \mathcal{L}_{(s)} \right]. \quad (\text{B.6})$$

We use the facts that the bulk Lagrangians $\mathcal{L}_{(b)}$, $\mathcal{L}_{(e)}$ do not have any dependence on the shape-field f , and the dependence of $\mathcal{L}_{(s)}$ only comes via the reparametrization invariant n_μ . For the sake of simplicity, we further assume that $\mathcal{L}_{(s)}$ is only dependent on n_μ and not on its derivatives, which is true for our analysis in the bulk of the paper. We can perform a f variation of W to get

$$\delta_f W = \int d^4x \sqrt{-\mathcal{G}} \tilde{\delta}(f) \left[\mathcal{L}_{(b)} - \mathcal{L}_{(e)} + \nabla_\mu \left(\mathcal{L}_{(s)} n^\mu + (\delta_\mu^\nu - n_\mu n^\nu) \frac{\partial \mathcal{L}_{(s)}}{\partial n_\mu} \right) \right] \frac{\delta f}{\sqrt{\nabla^\mu f \nabla_\mu f}}. \quad (\text{B.7})$$

This allows us to write the Young-Laplace equation directly as

$$\mathcal{L}_{(b)} - \mathcal{L}_{(e)} + \nabla_\mu \left(n^\mu \mathcal{L}_{(s)} + (\delta_\mu^\nu - n_\mu n^\nu) \frac{\partial \mathcal{L}_{(s)}}{\partial n_\nu} \right) = 0. \quad (\text{B.8})$$

On the other hand if we perform a restricted variation of W along $\vartheta^\mu = \vartheta n^\mu$ keeping f fixed one can check that we get

$$\begin{aligned}
 \delta_{\vartheta n^\mu} W &= \int d^4x \sqrt{-\mathcal{G}} \left[\left(\theta(f) \mathcal{L}_{(b)} + \theta(-f) \mathcal{L}_{(e)} + \tilde{\delta}(f) \mathcal{L}_{(s)} \right) \mathcal{G}^{\mu\nu} \nabla_\mu (\vartheta n_\nu) \right. \\
 &\quad \left. + \theta(f) \vartheta n^\mu \partial_\mu \mathcal{L}_{(b)} + \theta(-f) \vartheta n^\mu \partial_\mu \mathcal{L}_{(e)} + \tilde{\delta}(f) \vartheta n^\mu \partial_\mu \mathcal{L}_{(s)} - \tilde{\delta}(f) \frac{\partial \mathcal{L}_{(s)}}{\partial n_\mu} \delta_f n_\mu \right]. \quad (\text{B.9})
 \end{aligned}$$

We have used the fact that $\mathcal{L}_{(b)}$, $\mathcal{L}_{(e)}$ and $\mathcal{L}_{(s)}$ are scalars and transform accordingly. Note however that $\mathcal{L}_{(s)}$ also contains f which we are supposed to keep constant. To balance this we subtract the last term in (B.9). We can simplify this expression as

$$\delta_{\vartheta n^\mu} W = \int d^4x \sqrt{-\mathcal{G}} \vartheta \tilde{\delta}(f) \left[\mathcal{L}_{(b)} - \mathcal{L}_{(e)} + \nabla_\mu \left(\mathcal{L}_{(s)} n^\mu + (\delta_\mu^\nu - n_\mu n^\nu) \frac{\partial \mathcal{L}_{(s)}}{\partial n_\mu} \right) \right], \quad (\text{B.10})$$

which leads to the same Young-Laplace equation (B.8).

| Background Quantities | |
|--------------------------------|--|
| Metric & cov. derivative | $\mathcal{G}_{\mu\nu}, \quad (\mathcal{G} = \det \mathcal{G}_{\mu\nu}), \quad \nabla_\mu$ |
| Gauge field & strength | $\mathcal{A}_\mu, \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ |
| Levi-Civita tensor | $\epsilon^{\mu\nu\rho\sigma}: \epsilon^{0123} = \frac{1}{\sqrt{-\mathcal{G}}} (3+1), \quad \epsilon^{\mu\nu\rho}: \epsilon^{012} = \frac{1}{\sqrt{-\mathcal{G}}} (2+1)$ |
| Superfluid Quantities | |
| Fluid velocity | u^μ with $u^\mu u_\mu = -1$ |
| Temperature & chemical pot. | $T, \quad \mu, \quad (\nu = \mu/T)$ |
| Superfluid phase & velocity | $\phi, \quad \xi_\mu = -\partial_\mu \phi + \mathcal{A}_\mu, \quad (\zeta_\mu = \xi_\mu + (u^\nu \xi_\nu) u_\mu)$ |
| Superfluid potential | $\chi = -\xi^\mu \xi_\mu, \quad \hat{\chi} = -\zeta^\mu \zeta_\mu$ |
| Shape-field & normal vector | $f, \quad n_\mu = -\frac{1}{\sqrt{\nabla^\nu f \nabla_\nu f}} \partial_\mu f$ |
| Surface metric & derivative | $\tilde{\mathcal{G}}^{\mu\nu} = \mathcal{G}^{\mu\nu} - n^\mu n^\nu, \quad \tilde{\nabla}_\mu(\cdots) = \frac{1}{\sqrt{\nabla^\nu f \nabla_\nu f}} \nabla_\mu(\sqrt{\nabla^\nu f \nabla_\nu f} \cdots)$ |
| Surface fluid velocity | $\tilde{u}^\mu = u^\mu - (u^\nu n_\nu) n^\mu$ |
| Surface superfluid velocity | $\tilde{\xi}^\mu = \xi^\mu - (n^\nu \xi_\nu) n^\mu, \quad (\tilde{\zeta}^\mu = \zeta^\mu - (n^\nu \zeta_\nu) n^\mu)$ |
| Even surface scalars | $u^\mu n_\mu, \quad \tilde{\chi} = \tilde{\xi}^\mu \tilde{\xi}_\mu, \quad \lambda = n^\mu \xi_\mu$ |
| Distribution fn. & derivatives | $\theta(f), \quad \tilde{\delta}(f) = -n^\mu \partial_\mu \theta(f), \quad \tilde{\delta}'(f) = -n^\mu \partial_\mu \tilde{\delta}(f)$ |
| Odd surface velocity | $\bar{n}^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \xi_\rho n_\sigma (3+1), \quad \bar{n}^\mu = \epsilon^{\mu\nu\rho} u_\nu n_\rho (2+1)$ |
| Odd surface scalar (2 + 1) | $\bar{\lambda} = \epsilon^{\mu\nu\rho} n_\mu u_\nu \xi_\rho$ |
| Energy-momentum tensor | $T_{(b)}^{\mu\nu}$ (bulk), $T_{(s)}^{\mu\nu}$ (surface), $T_{(e)}^{\mu\nu}$ (exterior), $T^{\mu\nu}$ (full) |
| Charge current | $J_{(b)}^\mu$ (bulk), $J_{(s)}^\mu$ (surface), $J_{(e)}^\mu$ (exterior), J^μ (full) |
| Entropy current | $J_{(b)\text{ent}}^\mu$ (bulk), $J_{(s)\text{ent}}^\mu$ (sur.), $J_{(e)\text{ent}}^\mu$ (ext.), J_{ent}^μ (full) |
| Derivative corrections | $\Pi_{(b/s/e)}^{\mu\nu}, \quad \Upsilon_{(b/s/e)}^\mu, \quad \Upsilon_{(b/s/e)\text{ent}}^\mu$ |
| Equations of Motion | |
| Energy-mom. conservation | $\nabla_\mu T_{(b/e)}^{\mu\nu} = \mathcal{F}^{\nu\rho} J_{(b/e)}^\mu, \quad \tilde{\nabla}_\mu T_{(s)}^{\mu\nu} = \mathcal{F}^{\nu\rho} J_{(s)}^\mu + n_\mu (T_{(b)}^{\mu\nu} - T_{(e)}^{\mu\nu})$ |
| Charge conservation | $\nabla_\mu J_{(b/e)}^\mu = 0, \quad \tilde{\nabla}_\mu J_{(s)}^\mu = n_\mu (J_{(b)}^\mu - J_{(e)}^\mu)$ |
| Local second law | $\nabla_\mu J_{(b/e)\text{ent}}^\mu \geq 0, \quad \nabla_\mu J_{(s)\text{ent}}^\mu - n_\mu (J_{(b)\text{ent}}^\mu - J_{(e)\text{ent}}^\mu) \geq 0$ |
| Young-Laplace equation | $n_\nu \tilde{\nabla}_\mu T_{(s)}^{\mu\nu} = n_\nu \mathcal{F}^{\nu\rho} J_{(s)}^\mu + n_\mu n_\nu (T_{(b)}^{\mu\nu} - T_{(e)}^{\mu\nu})$ |
| f equation of motion | $u^\mu n_\mu = \gamma_{\text{diss}}$ |
| Josephson equation | $u^\mu \xi_\mu = \mu + \mu_{\text{diss}}$ |

Table 3. Quick reference guide for relativistic superfluid bubbles (Part I).

C Useful notations and formulae

In this appendix, we recollect some useful notations and formulae used throughout this paper.

Relativistic superfluids. We have given a list of useful definitions and relations for relativistic superfluids in tables 3 and 4. In equilibrium, the metric and gauge field can be

| Constitutive Relations | |
|---|---|
| $T_{(b)}^{\mu\nu} = (E + P)u^\mu u^\nu + P\mathcal{G}^{\mu\nu} + F\xi^\mu \xi^\nu + \Pi_{(b)}^{\mu\nu}, \quad J_{(b)}^\mu = Qu^\mu - F\xi^\mu + \Upsilon_{(b)}^\mu$ $T_{(s)}^{\mu\nu} = (\mathcal{E} - \mathcal{C})u^\mu u^\nu - \mathcal{C}\tilde{\mathcal{G}}^{\mu\nu} + \mathcal{F}\tilde{\xi}^\mu \tilde{\xi}^\nu + 2\mathcal{U}u^{(\mu}\bar{n}^{\nu)} + \Pi_{(s)}^{\mu\nu}, \quad J_{(s)}^\mu = \mathcal{Q}u^\mu - \mathcal{F}\tilde{\xi}^\mu + \mathcal{V}\bar{n}^\mu + \Upsilon_{(s)}^\mu$ $J_{(b)\text{ent}}^\mu = Su^\mu + \Upsilon_{(b)\text{ent}}^\mu, \quad J_{(s)\text{ent}}^\mu = \mathcal{S}u^\mu + \frac{1}{T}(\mathcal{U} - \mu\mathcal{V})\bar{n}^\mu + \Upsilon_{(s)\text{ent}}^\mu$ | |
| Pressure and surface tension | P (bulk), $\mathcal{Y} = -\mathcal{C}$ (surface) |
| Bulk densities | E (energy), Q (charge), S (entropy), F (superfluid den.) |
| Surface densities | \mathcal{E} (energy), \mathcal{Q} (charge), \mathcal{S} (entropy), \mathcal{F} (superfluid den.) |
| First order even bulk coeff. | f_1, f_2 |
| First order odd bulk coeff. | g_1, g_2 (3+1), $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ (2+1) |
| Surface inflow densities (3+1) | $\mathcal{U} = g_1, \mathcal{V} = g_2$ (3+1), $\mathcal{U} = \kappa_1, \mathcal{V} = \kappa_2$ (2+1) |
| Euler relation | $E + P = TS + \mu Q$ (bulk), $\mathcal{E} - \mathcal{Y} = TS + \mu\mathcal{Q}$ (surface) |
| Bulk first law | $dE = TdS + \mu dQ - \frac{1}{2}Fd\chi$ |
| Surface first law (3+1) | $d(\mathcal{E} - \lambda f_1) = Td(\mathcal{S} - \frac{\lambda}{T}(f_1 - \mu f_2)) + \mu d(\mathcal{Q} - \lambda f_2) - \frac{1}{2}\mathcal{F}d\tilde{\chi}$ |
| Surface first law (2+1) | $d(\mathcal{E} - \lambda f_1 + \bar{\lambda}\kappa_4) = Td(\mathcal{S} - \frac{\lambda}{T}(f_1 - \mu f_2) + \frac{\bar{\lambda}}{T}(\kappa_3 - \mu\kappa_4))$ $+ \mu d(\mathcal{Q} - \lambda f_2 + \bar{\lambda}\kappa_4) - \frac{1}{2}\mathcal{F}d\tilde{\chi}$ |

Table 4. Quick reference guide for relativistic superfluid bubbles (Part II).

dimensionally reduced in a Kaluza-Klein framework as

$$\mathcal{G}_{\mu\nu} = \begin{pmatrix} -e^{2\sigma} & -e^{2\sigma}a_j \\ -e^{2\sigma}a_i & -e^{2\sigma}a_i a_j + g_{ij} \end{pmatrix}, \quad \mathcal{G}^{\mu\nu} = \begin{pmatrix} -e^{-2\sigma} + a^k a_k & -a^j \\ -a^i & g^{ij} \end{pmatrix}, \quad (\text{C.1})$$

$$\mathcal{A}_\mu = \begin{pmatrix} A_0 \\ A_i + a_i A_0 \end{pmatrix}, \quad \mathcal{A}^\mu = \begin{pmatrix} -e^{-2\sigma}A_0 - a_j A^j \\ A^i \end{pmatrix}. \quad (\text{C.2})$$

Let W be a partition function for relativistic superfluids with a surface, written as a functional of the background fields σ , a_i , A_0 , A_i , g_{ij} , the Goldstone boson ϕ and the shape-field f . Varying it, we can read out the components of the energy-momentum tensor and charge current as,

$$\begin{aligned} T_{00} &= -\frac{T_0 e^\sigma}{\sqrt{g}} \frac{\delta W}{\delta \sigma}, & T_0^i &= \frac{T_0 e^{-\sigma}}{\sqrt{g}} \left(\frac{\delta W}{\delta a_i} - A_0 \frac{\delta W}{\delta A_i} \right), & T^{ij} &= 2 \frac{T_0 e^{-\sigma}}{\sqrt{g}} \frac{\delta W}{\delta g_{ij}}, \\ J_0 &= -\frac{T_0 e^\sigma}{\sqrt{g}} \frac{\delta W}{\delta A_0}, & J^i &= \frac{T_0 e^{-\sigma}}{\sqrt{g}} \frac{\delta W}{\delta A_i}. \end{aligned} \quad (\text{C.3})$$

Here T_0 is the inverse radius of the Euclidean time circle.

Null/Galilean superfluids. In a similar spirit (see [10] for details), let W be a partition function for null superfluids with a surface, written in terms of the background fields σ , a_i , A_0 , A_i , B_0 , B_i , g_{ij} , the Goldstone boson ϕ and the shape-field f . Upon variation, it gives

various components of the energy-momentum tensor and charge current respectively as

$$\begin{aligned}
T_{--} &= \frac{T_0}{\sqrt{g}} \frac{\delta W}{\delta B_0}, & T_{0-} &= -\frac{T_0}{\sqrt{g}} \left(\frac{\delta W}{\delta \sigma} + B_0 \frac{\delta W}{\delta B_0} \right), \\
T_{-}^i &= -\frac{T_0 e^{-\sigma}}{\sqrt{g}} \left(\frac{\delta W}{\delta B_i} - \frac{\delta W}{\delta A_i} \right), & T_0^i &= \frac{T_0 e^{-\sigma}}{\sqrt{g}} \left(\frac{\delta W}{\delta a_i} - A_0 \frac{\delta W}{\delta A_i} \right), & T^{ij} &= \frac{2T_0 e^{-\sigma}}{\sqrt{g}} \frac{\delta W}{\delta g_{ij}}, \\
J_{-} &= -\frac{T_0}{\sqrt{g_3}} \frac{\delta W}{\delta A_0}, & J^i &= \frac{T_0 e^{-\sigma}}{\sqrt{g_3}} \frac{\delta W}{\delta A_i}.
\end{aligned} \tag{C.4}$$

Note that the components T_{00} and J_0 is not determined by the partition function. In fact, these two components are “unphysical” as they do not enter the respective conservation laws. The formulae (C.4) can also be recasted directly into a null reduced Galilean language

$$\begin{aligned}
\rho &= \frac{T_0}{\sqrt{g}} \frac{\delta W}{\delta B_0}, & \rho^i &= \frac{T_0 e^{-\sigma}}{\sqrt{g}} \left(\frac{\delta W}{\delta B_i} - \frac{\delta W}{\delta A_i} \right), & t^{ij} &= \frac{2T_0 e^{-\sigma}}{\sqrt{g}} \frac{\delta W}{\delta g_{ij}}, \\
\epsilon &= -\frac{T_0 e^{-\sigma}}{\sqrt{g}} \frac{\delta W}{\delta \sigma}, & \epsilon^i &= \frac{T_0 e^{-2\sigma}}{\sqrt{g}} \left(-\frac{\delta W}{\delta a_i} + (A_0 - B_0) \frac{\delta W}{\delta A_i} + B_0 \frac{\delta W}{\delta B_i} \right), \\
q &= \frac{T_0}{\sqrt{g}} \frac{\delta W}{\delta A_0}, & q^i &= \frac{T_0 e^{-\sigma}}{\sqrt{g}} \frac{\delta W}{\delta A_i}.
\end{aligned} \tag{C.5}$$

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